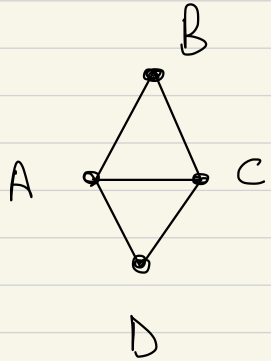
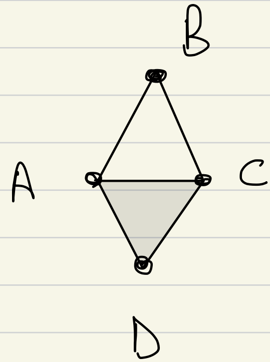
CPSC 531F March 28, 2025

Last time:

various





K^1 K^2 K^3

$$H_1(K^1) \longrightarrow H_1(K^2) \longrightarrow H_1(K^3)$$

Edelsbrunner, Letscher, Zomorodian:

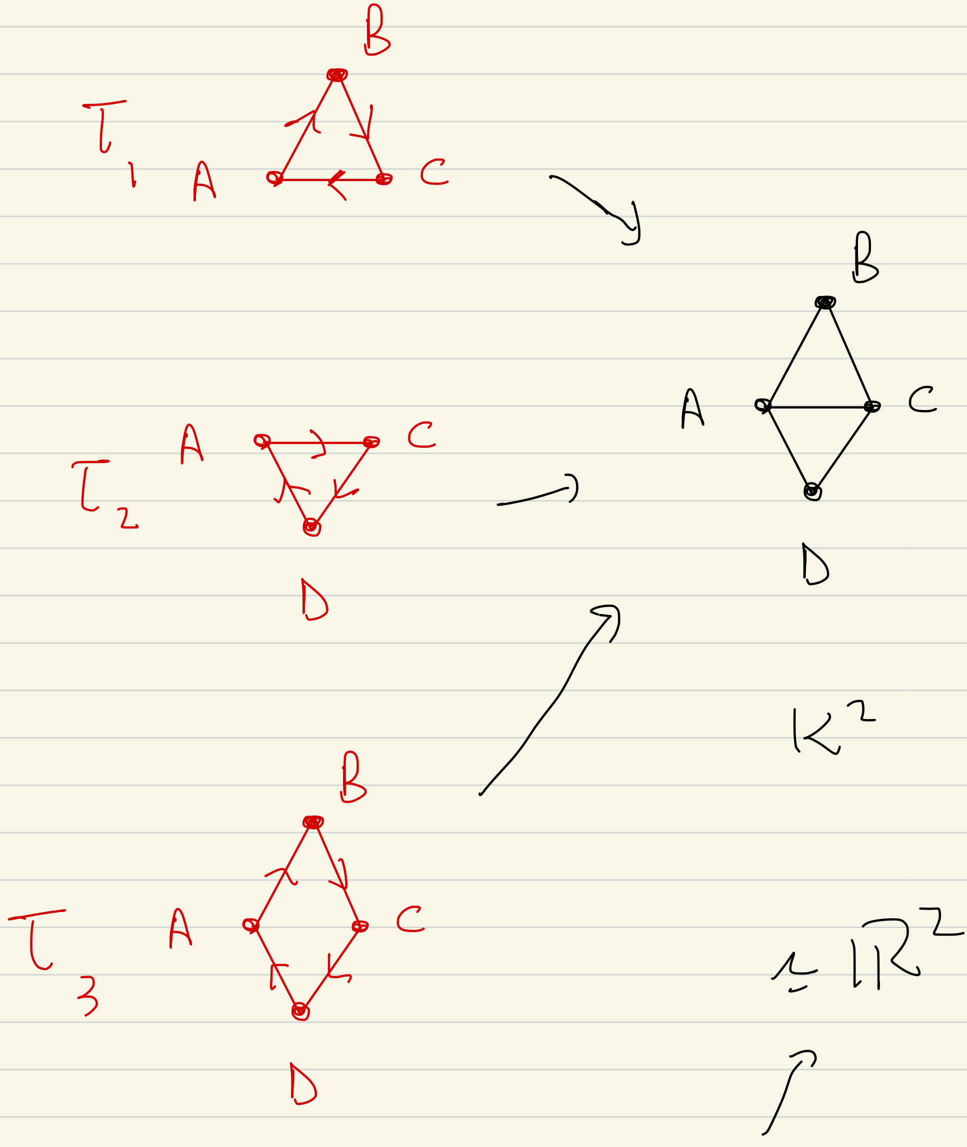
"simplify" the H_1 "evolution"

"persistance"

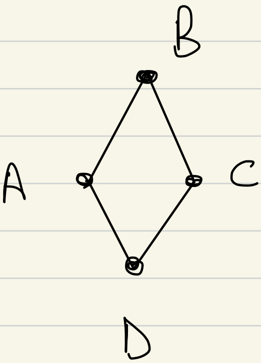
New as of March 27!

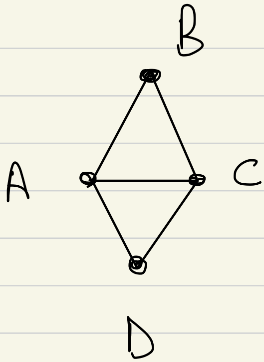
- A guide to the homework & handouts (e.g. which homework problems done in class)

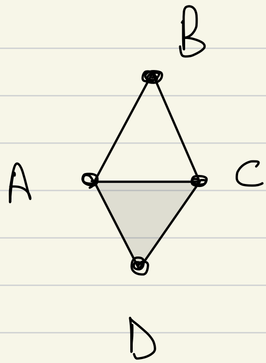
- Revised handout to include "Barcode" article



Note! $T_1 + T_2 = T_3$ in $H_1(K^2)$



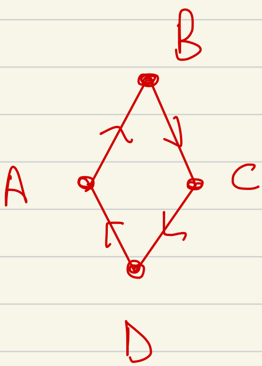


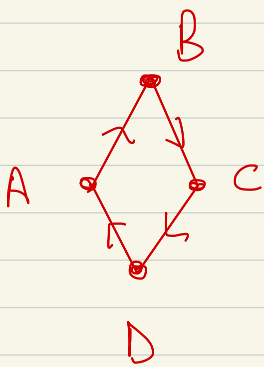


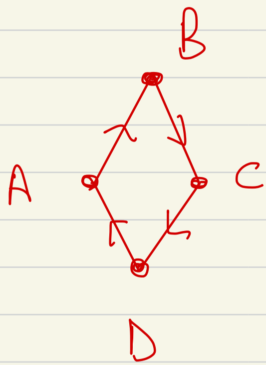
K^1 $\qquad$ K^2 $\qquad$ K^3

$$H_1(K^1) \longrightarrow H_1(K^2) \longrightarrow H_1(K^3)$$

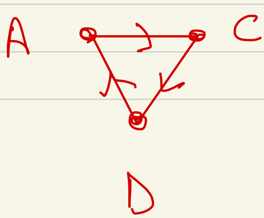
$\simeq \mathbb{R}$ $\qquad$ $\simeq \mathbb{R}^2$ $\qquad$ $\simeq \mathbb{R}$



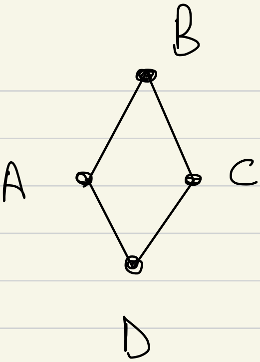


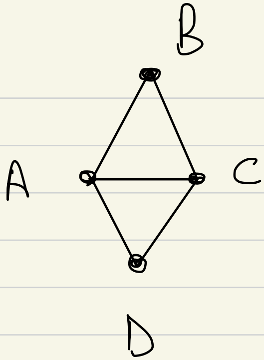


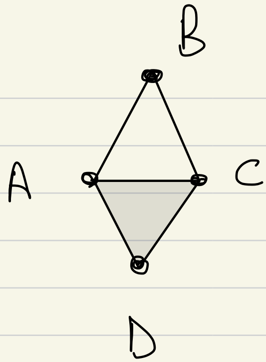
τ_3



τ_2



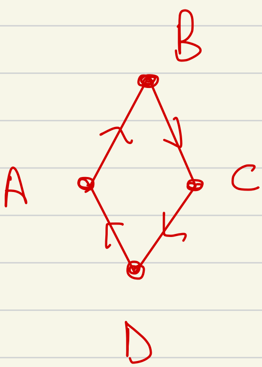


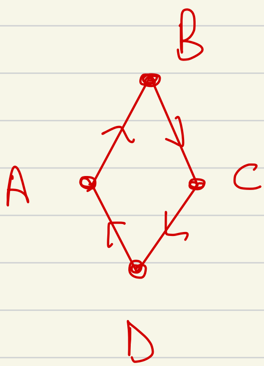


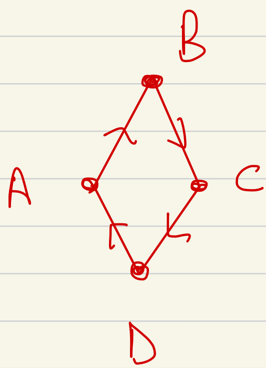
K^1 $\qquad$ K^2 $\qquad$ K^3

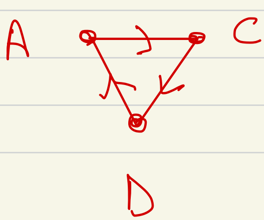
$$H_1(K^1) \longrightarrow H_1(K^2) \longrightarrow H_1(K^3)$$

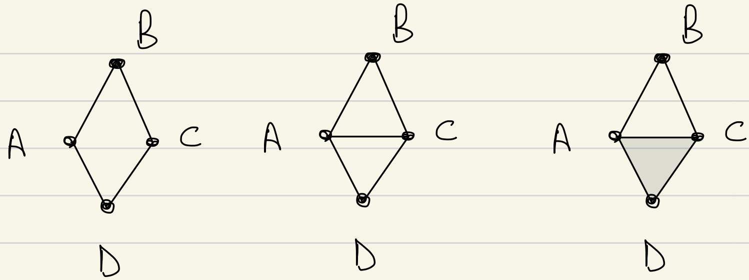
$\cong \mathbb{R}$ $\qquad$ $\cong \mathbb{R}^2$ $\qquad$ $\cong \mathbb{R}$









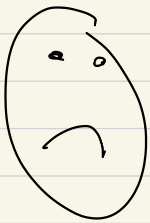


K^1 $\qquad$ K^2 $\qquad$ K^3

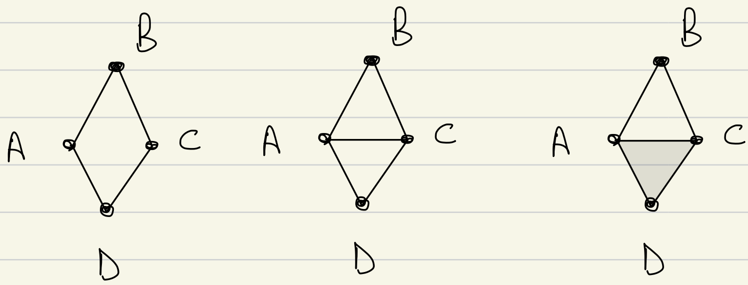
$$\tau_3 \circlearrowright \quad \mathbb{R} \to \mathbb{R} \to \mathbb{R}$$

$$\tau_2 \rightleftarrows \quad 0 \to \mathbb{R} \to 0$$

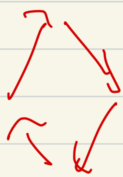
$$H_1(K^1) \to H_1(K^2) \to H_1(K^3)$$

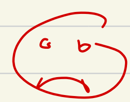


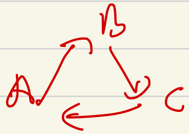
τ_3, τ_1



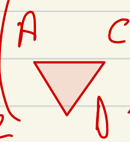
K^1 $\quad$ K^2 $\quad$ K^3

τ_3  $\mathbb{R} \to \mathbb{R} \to \mathbb{R}$



τ_1 

$0 \qquad \mathbb{R} \to \mathbb{R}$

in $H_1(K^3)$, $\tau_3 = \tau_1 = \tau_2 = \partial_2\Big($  $\Big)$

$\tau_3 = \tau_1$ in $H_1(K^2)$

Abstractly

$$H_1(K^1) \to H_1(K^2) \to H_1(K^3)$$

$$\wr \qquad\qquad \wr \qquad\qquad \wr$$

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} V^2$$

Take an arbitrary "string of vector spaces"

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^n$$

can we decompose "nice"

Nice situation with in the last picture

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \overline{V}^2$$

Example:

$$\left.\begin{array}{ll}
\color{red}{\text{this} \to} & \mathbb{R} \xrightarrow{id} \mathbb{R} \xrightarrow{id} \mathbb{R} \\
& \mathbb{R} \longrightarrow 0 \longrightarrow 0 \\
\color{red}{\oplus} & \\
\color{red}{\text{this} \to} & 0 \longrightarrow \mathbb{R} \longrightarrow 0 \\
& 0 \longrightarrow 0 \longrightarrow \mathbb{R} \\
& \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0 \\
& 0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}
\end{array}\right\} \text{"simplest"}$$

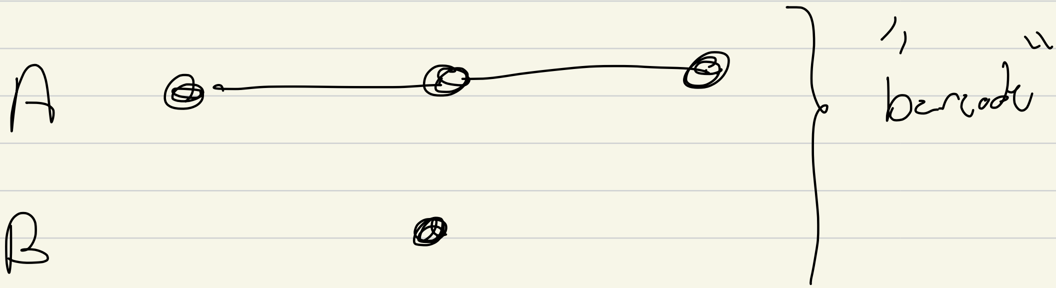
$$\mathbb{R} \oplus 0 \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus 0$$

$$\cong \mathbb{R} \qquad \cong \mathbb{R}^2 \qquad \cong \mathbb{R}$$

A: $\mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}$

B: $0 \longrightarrow \mathbb{R} \longrightarrow 0$

$$\mathbb{R}\{A\} \longrightarrow \mathbb{R}\{A, B\} \longrightarrow \mathbb{R}\{A\}$$



$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V_n$$

string of vector spaces (over $\mathbb{R}$)

of length $n+1$

An "$\underline{(i,j)\text{-bar}}$"

$$\cdots 0 \to V^i \xrightarrow{id} V^{i+1} \xrightarrow{id} \cdots \xrightarrow{id} V^j \to 0 \cdots 0$$

$$V^i = \mathbb{R}, \quad V^{i+1} = \mathbb{R}, \quad V^j = \mathbb{R}$$

i.e., $i \leq j$

$$V^i = V^{i+1} = \cdots = V^j \cong \mathbb{R}$$

$$V^{i-1}, V^{i-2}, \ldots, \quad V^{j+1}, V^{j+2}, \ldots = 0$$

and maps

$$V^i \xrightarrow{id} V^{i+1} \xrightarrow{id} \cdots \xrightarrow{id} V^j$$

$id = \text{identity}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$

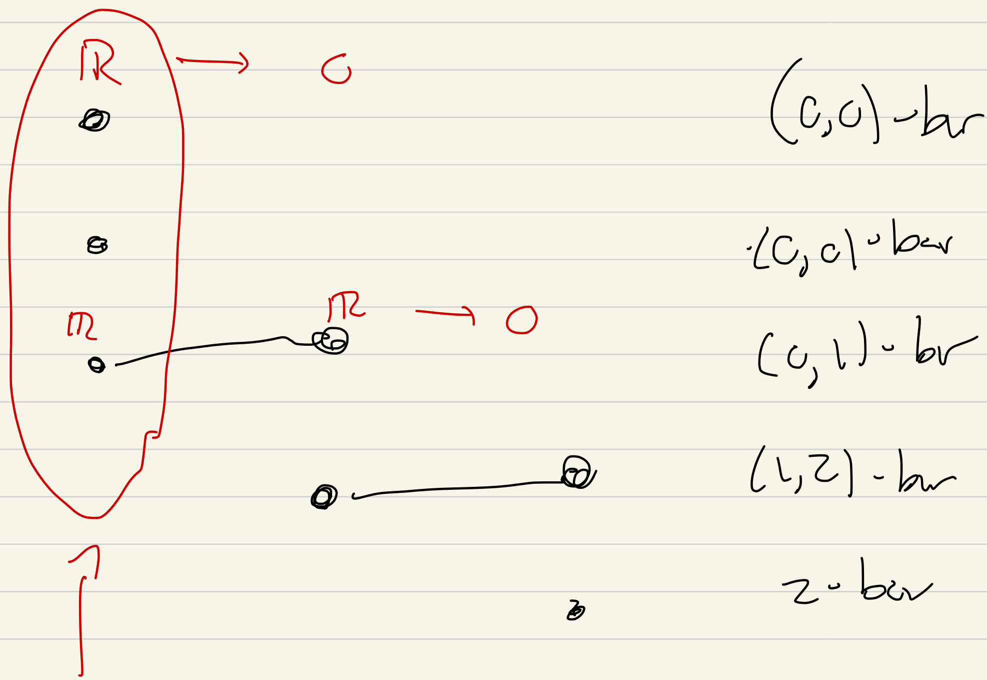
Thm says: any string of vector spaces $V^0 \xrightarrow{\mathcal{L}^0} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^n$ can be written as a direct sum of bars.

Also # (i,j) bars in decomposition is unique for each $i \le j$.

Intuitively:

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} V^2$$

say we write



these 3 dots represent a basis for V^0

If

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} V^2 \to \cdots$$

$$\hat{V}^0 \xrightarrow{\hat{\mathcal{L}}^0} \hat{V}^1 \xrightarrow{\hat{\mathcal{L}}^1} \hat{V}^2 \to \cdots$$

direct sum

$$\underbrace{V^0 \oplus \hat{V}^0}_{\text{block}} \xrightarrow[\text{block diagonal}]{\mathcal{L}^0 \oplus \hat{\mathcal{L}}_0} \underbrace{V^1 \oplus \hat{V}^1}_{\text{block}} \to \cdots$$

$$\left[\begin{array}{c|c} \mathcal{L}^0 & \\ \hline & \hat{\mathcal{L}}_0 \end{array}\right]$$

Similarly can direct sum of any finite number of strings of vector spaces length n.

Map (morphism) of strings:

$$
\begin{tikzcd}
V^0 \arrow[r, "\mathcal{L}^0"] \arrow[d, "m^0"'] & V^1 \arrow[r, "\mathcal{L}^1"] \arrow[d, "m^1"] & V^2 \arrow[r] \arrow[d, "m^2"] & \cdots \\
\tilde{V}^0 \arrow[r, "\tilde{\mathcal{L}}^0"] & \tilde{V}^1 \arrow[r] & \cdots &
\end{tikzcd}
$$

diagram commutes.

Isomorphism: $m^0, m^1, \ldots$ are isomorphism (bijective, linear)

Practically:

$$\mathbb{R} \to \mathbb{R} \to 0 \qquad A$$

$$\oplus$$

$$\mathbb{R} \to \mathbb{R} \to 0 \qquad B$$

$$\oplus$$

$$\mathbb{R} \to \mathbb{R} \to \mathbb{R} \qquad C$$

$$\oplus$$

$$0 \to \mathbb{R} \to 0 \qquad D$$

$$\mathbb{R}\{A,B,C\} \hookrightarrow \mathbb{R}\{A,B,C,D\} \to \mathbb{R}\{C\}$$

isomorphism

$$V^0 \xrightarrow{\mathcal{L}^{10}} V^1 \xrightarrow{\mathcal{L}^{0}} V^2$$

$$0 \longrightarrow 0$$

$$V^{1,A} = \mathcal{L}^0(V^{0,A})$$

$$0 = \mathcal{L}^1(V^{1,A})$$

$$\boxed{V^{0,A} \xrightarrow{\mathcal{L}^0} V^{1,A} \xrightarrow{\mathcal{L}^1} 0}$$

$$V^{0,B} \xrightarrow{\mathcal{L}^0} V^{1,B} \longrightarrow 0$$

$$V^{0,C} \longrightarrow V^{1,C} \longrightarrow V^{2,C}$$

$$0 \longrightarrow V^{1,D} \longrightarrow 0$$

basis

$$V^0 \qquad V^1 \qquad V^2$$

so

$V^{0,A}, V^{0,B}, V^{0,C}$ basis for V^0

$V^{1,A}, V^{1,B}, V^{1,C}, V^{1,D}$ " " V^1

ETC.

Example

$$V^0 \longrightarrow V^1 \longrightarrow V^2$$

$$\mathbb{R}^2 \xrightarrow{\text{id}} \mathbb{R}^2 \xrightarrow{\text{id}} \mathbb{R}^2$$

continued

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2025 \end{bmatrix} \in \mathbb{R}^2 \quad , \text{basis}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbb{R} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbb{R} \longmapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbb{R}$$

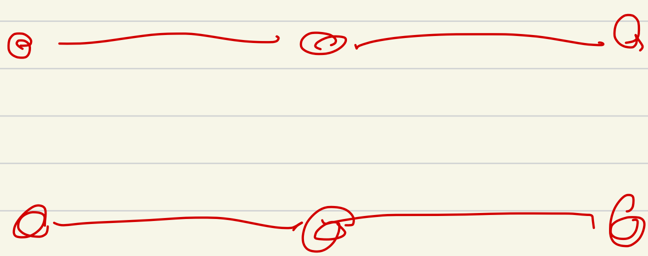
$\oplus$

$$\begin{bmatrix} 3 \\ 2025 \end{bmatrix}\mathbb{R} \rightarrow \begin{bmatrix} 3 \\ 2025 \end{bmatrix}\mathbb{R} \longrightarrow \cancel{\begin{bmatrix} 50 \\ -1 \end{bmatrix}\mathbb{R}}$$

$$\begin{pmatrix} 3 \\ 2025 \end{pmatrix}\cancel{t \longrightarrow \begin{bmatrix} 50 \\ \cdot 1 \end{bmatrix}(t/17)}$$

$$\begin{bmatrix}3\\2025\end{bmatrix}\mathbb{R} \to \begin{bmatrix}3\\2025\end{bmatrix}\mathbb{R} \to \begin{bmatrix}3\\2025\end{bmatrix}\mathbb{R}$$

2 bars



most persistent bar (if exist)

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

$$V^0 \qquad V^1 \qquad\qquad V^n$$

$$V^0 \longrightarrow \mathcal{L}^0(V^0) \longrightarrow \mathcal{L}^{n-1}\cdots\mathcal{L}^1\mathcal{L}^0(V^0)$$