CPSC 531F March 31, 2025

Today:

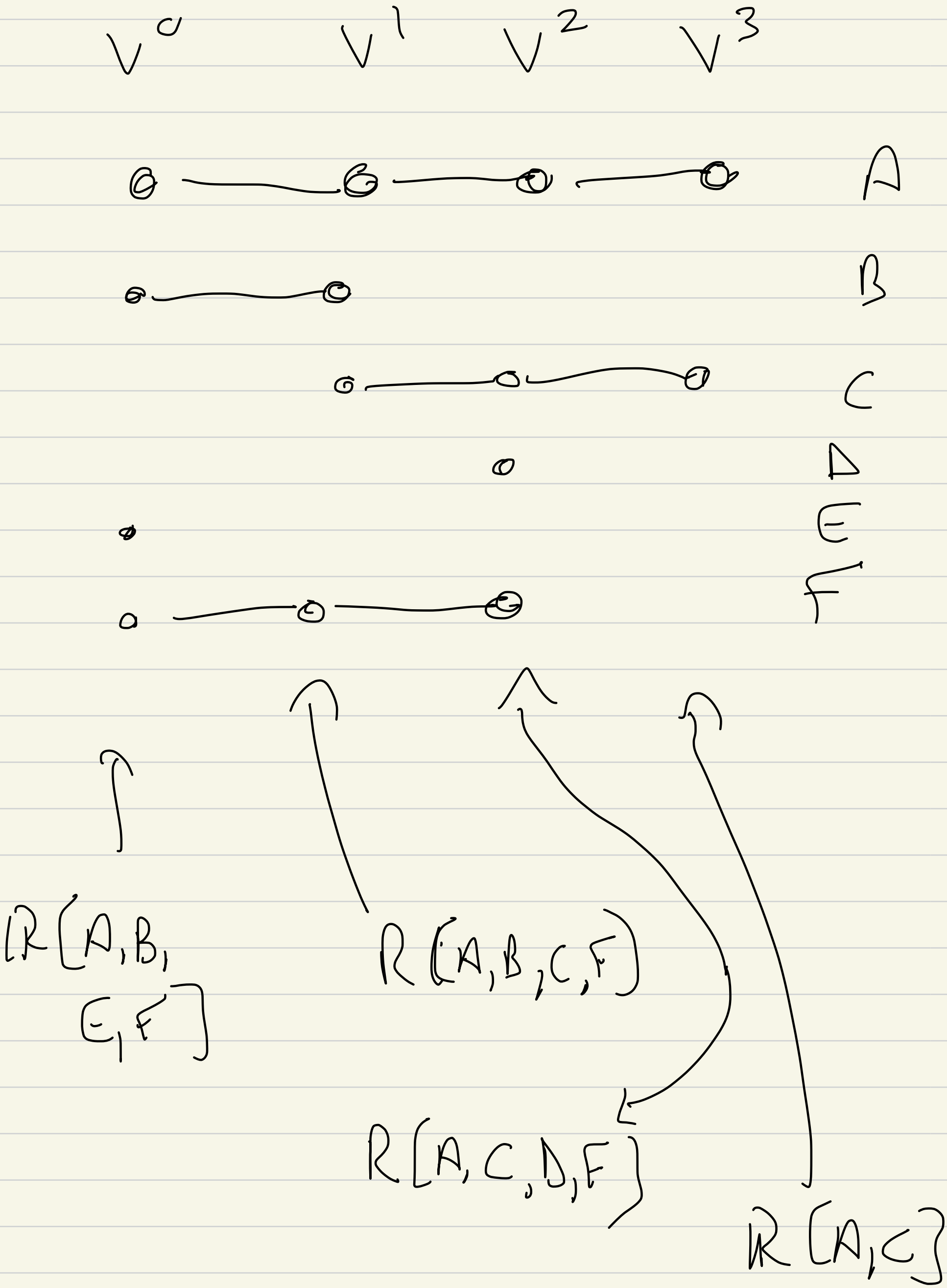
Start: abstract setting:

a string of vector spaces of length $n+1$:

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} V^n$$

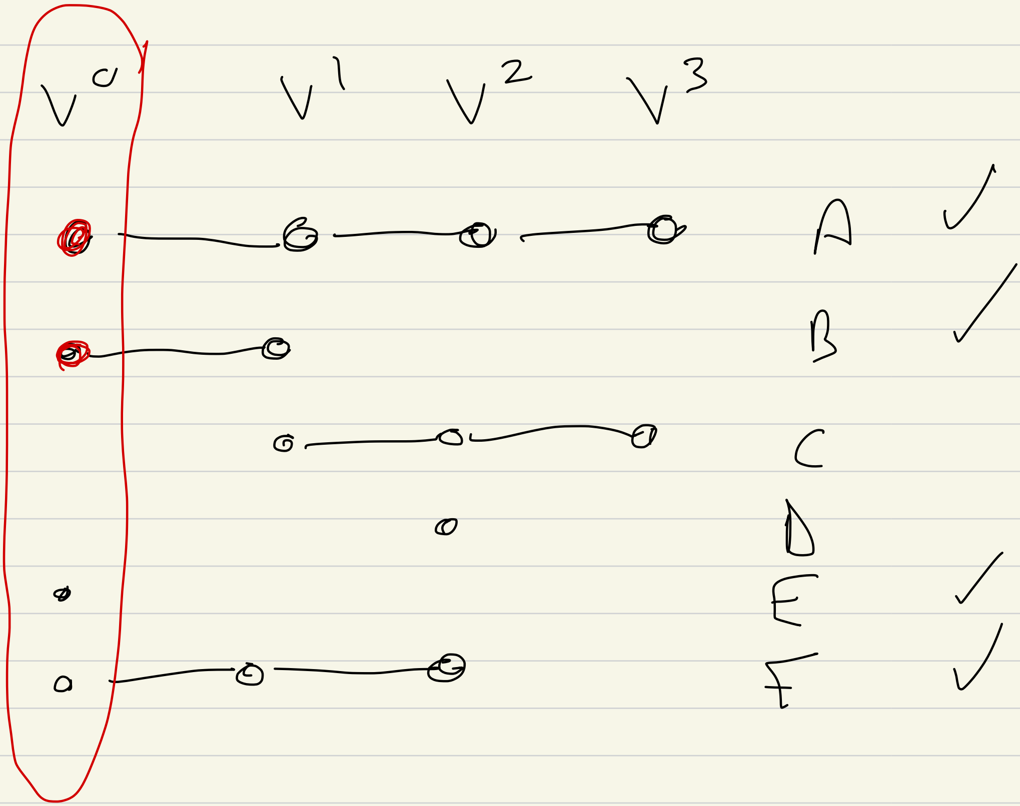
Goal: decompose into "bars"

Idea: Imagine that there is a "bar decomposition":


V0
V1
V2
V3
A
B
C
D
E
F
R[A,B,E,F]
R[A,B,C,F]
R[A,C,D,F]
R[A,C]

Imagine we look at all bars

(0, q) - bars



These must a basis for V^0

These are V'

$$v^{\{0,1\}} \xrightarrow{\mathcal{L}^0} \mathcal{L}^0\left(v^{\{0,1\}}\right)$$

$$v^{\{0,2\}} \longrightarrow \mathcal{L}^0\left(v^{\{0,2\}}\right)$$

$$\vdots$$

$v^{\{0,3\}}$

$v^{\{0,4\}}$

$$\underbrace{}_{\text{span}}$$

has to lie in

$$\mathcal{L}^0\left(V^0\right)$$

Consider:

Step 1:

$$\begin{array}{ccccccc}
\tilde V^0 & \longrightarrow & \mathcal{L}^0(\tilde V^0) & \longrightarrow & \mathcal{L}^1\mathcal{L}^0(\tilde V^0) & \longrightarrow & \mathcal{L}^2\mathcal{L}^1\mathcal{L}^0(\tilde V^0) \\
\| & & \cap & & \cap & & \cap \\
\tilde V^0 & & \tilde V^1 & & \tilde V^2 & & \tilde V^3
\end{array}$$

Notation:

$\tilde V^{i \to j}$ = image in $\tilde V^j$ of $\tilde V^i$

$$\tilde V^i \xrightarrow{\mathcal{L}^i} \tilde V^{i+1} \xrightarrow{\mathcal{L}^{i+1}} \cdots \xrightarrow{\mathcal{L}^{j-1}} \tilde V^j$$

$$\mathcal{L}^{j-1} \circ \cdots \circ \mathcal{L}^{i+1} \circ \mathcal{L}^i : \tilde V^i \longrightarrow \tilde V^j$$

$$\underbrace{\mathcal{L}^{i\to j}}_{} : \bar{V}^i \longrightarrow \bar{V}^j \qquad (i < j)$$

denotes $\mathcal{L}^{j-1} \circ \cdots \circ \mathcal{L}^{i+1} \circ \mathcal{L}^i$

$$\mathcal{L}^{i\to i} : \bar{V}^i \longrightarrow \bar{V}^i$$

identity

"empty composition"

$$\bar{V}^{i\to j} = \text{image of } \bar{V}^i \text{ in } \bar{V}^j$$

$$= \mathcal{L}^{i\to j}(\bar{V}^i)$$

$$= \mathrm{Image}(\mathcal{L}^{i\to j})$$

Step 1:

$$\begin{tikzcd}
\bar V^0 \arrow[r] & \bar V^{0\to 1} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \bar V^{0\to 2} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r] & \bar V^{0\to n} \arrow[d, phantom, "\shortparallel"] \\
 & \mathcal{L}^{0\to 1}(\bar V^0) \arrow[d, phantom, "\cap"] & \mathcal{L}^{0\to 2}(\bar V^0) \arrow[d, phantom, "\cap"] & \cdots & \mathcal{L}^{0\to n}(\bar V^0) \arrow[d, phantom, "\cap"] \\
 & \bar V^1 & \bar V^2 & \cdots & \bar V^n
\end{tikzcd}$$

First: pick a basis for

$$m_{0\to n} \overset{def}{=} \dim\left(\mathcal{L}^{0\to n}(\bar V^0)\right)$$

$$= \dim\left(\bar V^{0\to n}\right)$$

"Sweeping forward"

$$V^0 \xrightarrow{\mathcal{L}^{0\to n}} \bar{V}^{0\to n}$$

$= \text{Image}(\mathcal{L}^{0\to n})$

$\bar{V}^{0\to n}$

$\uparrow$

$\dim = m_{0\to n}$

$\uparrow$

basis

$w^{0,1}, \dots, w^{0,m_{0,n}}$

$\uparrow$

for $m_{0\to n}$

Choose

$v^{0,1}, \dots, v^{0,m_{0,n}} \in \bar{V}^0$

s.t. $\mathcal{L}^{0\to n}(v^{0,i}) = w^{0,i}$

$$\underbrace{v^{0,i} \mapsto \mathcal{L}^{0\to 1} v^{0,i} \mapsto \mathcal{L}^{0\to 2} v^{0,i} \mapsto \cdots \mapsto \underset{\neq 0}{\mathcal{L}^{0\to n} v^{0,i}}}_{\text{gives a } [0,n)\text{-bar}}$$

$1 \leq i \leq m_{0,n}$

Claim: Look at these $m_{0,n}$ bars

$1 \leq i \leq m_{0,n}$

$$\begin{array}{ll}
i=1 & \bullet \text{———} \bullet \text{———} \bullet \;\cdots\; \text{———} \bullet \\
i=2 & \bullet \text{———} \bullet \text{———} \bullet \;\cdots\; \text{———} \bullet \\
\vdots & \\
 & V^0 \longrightarrow V^1 \longrightarrow \cdots \longrightarrow V^n
\end{array}$$

Claim: $0 \le j \le n$ and

$$\left.\begin{array}{c} \mathcal{L}^{0\to j}(v^{0,1}) \\ \vdots \\ \mathcal{L}^{0\to j}(v^{0,m_{0,n}}) \end{array}\right\} \text{ are linearly independent in } \bar{V}^j$$

$$\cap$$

$$\bar{V}^j$$

Why? Say

$$\alpha_1 \mathcal{L}^{0\to j}(v^{0,1}) + \dots + \alpha_{m_{0,n}} \mathcal{L}^{0\to j}(v^{0,m_{0,n}}) = 0$$

Apply $\mathcal{L}^{j \to n}$ to both sides

$$\alpha_1 \underbrace{\mathcal{L}^{j \to n}\left(\mathcal{L}^{c \to j} V^{c,1}\right)}_{\substack{\left(\mathcal{L}^{j \to n} \mathcal{L}^{c \to j}\right) V^{c,1} \\ \left(\mathcal{L}^{c \to n}\right) V^{c,1} \\ W^{c,1}}} + \cdots$$

We get

$$\alpha_1 W^{c,1} + \cdots + \alpha_{m_{c,n}} W^{c, m_{c,n}} = 0$$

So $\alpha_1 = \cancel{\alpha_2} = \cdots = 0$ since $W^{c,i}$ basis

End of 1st step: we found

$(0,n)$-bars

$$V^{0,1} \longmapsto \mathcal{L}^{0\to 1} V^{0,1} \longmapsto \mathcal{L}^{0\to 2} V^{0,1} \longrightarrow$$

$$\cdots \longrightarrow \mathcal{L}^{0\to n} V^{0,1} = \underset{\substack{\uparrow \\ \text{non-zero}}}{W^{0,1}}$$

\# $(0,n)$-bars

$$m_{0,n} = \dim\left({}'V^{0\to n}\right)$$

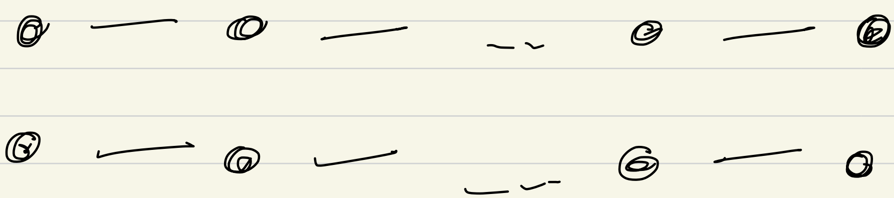
Claim: This is good...

2^{nd} Step:

$$V^{c,1} \to \mathcal{L}^{c,1} V^{c,1} \to \cdots \to \mathcal{L}^{c,n-1} V^{c,1} \to \mathcal{L}^{c,n} V^{c,1}$$

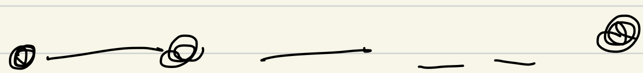
$$\vdots$$

$$V^{c,m_{c,r}} \to \cdots \qquad \mathcal{L}^{c,n} V^{c,m_{c,n}}$$



$V^0 \qquad V^1 \qquad V^{n-1} \qquad V^n$

Now look for



$(c, n-1)$ bars

the vector here

$\mathcal{L}^{n-1}$

0

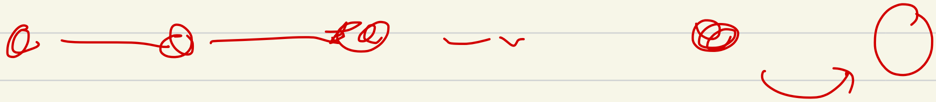
gives a basis --- (c,n)-basis

How: look

$$V^{c} \to V^{c\to 1} \to \bar{V}^{c\to 2} \to \dots \bar{V}^{c\to n-1} \to V^{c,n}$$

restriction of $\mathcal{L}^{n-1}$ to $\bar{V}^{c,n-1}$

any $(c,n-1)$-basis



look at $\bar{V}^{c,n-1} \xrightarrow{\mathcal{L}^{n-1}} \bar{V}^{c,n}$

ker of $\mathcal{L}^{n-1}$ in

$$\ker \mathcal{L}^{n-1}$$

in

$$V^{c,n-1} \xrightarrow{\mathcal{L}^{n-1}} V^{c,n}$$

$$\tilde{\omega}^{c,\bar{i}}$$

$$m_{c,n}+1 \leq \bar{i} \leq m_{c,n-1}$$

$$m_{c,n} = \dim(V^{c,n})$$

$$m_{c,n-1} = \dim(V^{c,n-1})$$

$$\bar{V}^{c,n-1} \twoheadrightarrow \bar{V}^{c,n}$$

$$\dim(\ker(\mathcal{L}^{n-1}|_{\bar{V}^{c,n-1}})) = m_{c,n-1} - m_{c,n}$$