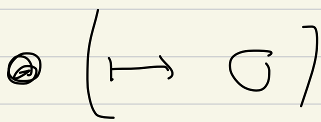
CPSC 531F April 2, 2025

- Homework: Hand in what you can by April 23; you can always hand in more homework later

$$\bar{V}^0 \xrightarrow{\mathcal{L}^0} \bar{V}^1 \xrightarrow{\mathcal{L}^1} \bar{V}^2 \longrightarrow \cdots$$

Say

$$\bar{V}^0 \longrightarrow \bar{V}^1 \longrightarrow \bar{V}^2$$

A 

B 

C 

$$\mathbb{R}[A,B,C] \to \mathbb{R}[B,C] \to \mathbb{R}[C]$$

$$\mathbb{R}\{A,B,C\}$$
$$= \{\alpha A + \beta B + \gamma C \mid \alpha, \beta, \gamma \in \mathbb{R}\}$$

(0,0)-bar

$$\tilde{V}^0 \qquad \tilde{V}^1 \qquad \tilde{V}^2$$

$$v \longmapsto 0 \longmapsto 0 \qquad \text{(0,0)-bar}$$

$\uparrow$

must live in $\mathrm{Ker}\left(\tilde{V}^0 \xrightarrow{\mathcal{L}^0} \tilde{V}^1\right)$

So

$$v = \alpha A, \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0$$

(0,2)-bar

$$C \quad \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$C \longmapsto C \longmapsto C$$

$$17C \longmapsto 17C \longmapsto 17C$$

Another $(c,2)$-bw that's not a scalar times C !

$$\longmapsto C+2B \longmapsto C$$

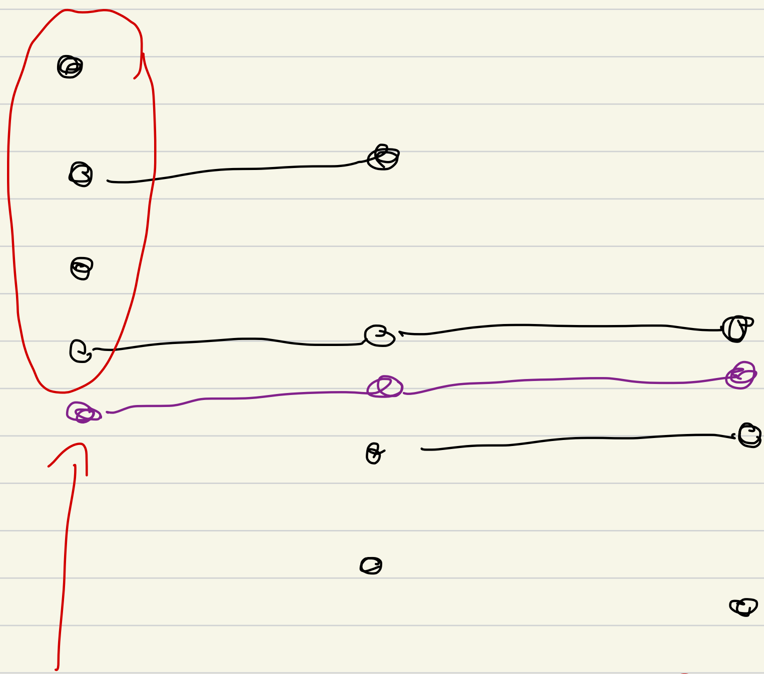
$$C+2B-3.1A$$

$$\tilde{V}^C \longmapsto \tilde{V}^1 \longrightarrow \tilde{V}^2$$

Algorithm to find bar decomposition:

"forward sweep"

$$\vec{V}^0 \longrightarrow \vec{V}^1 \longrightarrow \vec{V}^2$$



bars that start in $\vec{V}^0$

represents basis for $\vec{V}^0$

0th phase:

determine all $(0,q)$-bars

for $q = n, n-1, n-2, \ldots, 0$

(in this order)

Assuming a bar-decomposition exists

$$\bigoplus_{q=0}^{n} \left(\text{all } (0,q)\text{-bar}\right)$$

$$\begin{tikzcd}
\bar{V}^0 \arrow[r, "\mathcal{L}^0"] \arrow[d, equal] & \mathcal{L}^0(\bar{V}^0) \arrow[r, "\mathcal{L}^1"] \arrow[d, phantom, "\cap"] & \mathcal{L}^1(\mathcal{L}^0 \bar{V}^0) \arrow[r] \arrow[d, phantom, "\cap"] & \cdots \\
\bar{V}^0 & \bar{V}^1 & V^2 &
\end{tikzcd}$$

On the phase we look at

$$\bar{V}^{0} \longrightarrow \bar{V}^{0\to 1} \longrightarrow \bar{V}^{0\to 2} \longrightarrow \cdots$$

$$\mathcal{L}^{i\to j} : \bar{V}^{i} \to \bar{V}^{j} \qquad i \leq j$$

$$\parallel$$

$$\mathcal{L}^{j-1}_{0} \cdots \circ \mathcal{L}^{i+1} \circ \mathcal{L}^{i}$$

$$\bar{V}^{i\to j} = \text{Image of } \left(\mathcal{L}^{i\to j}\right)$$

$$= \mathcal{L}^{i\to j}\left(\bar{V}^{i}\right)$$

$$m_{i,j} \stackrel{\text{def}}{=} \dim\left(\bar{V}^{i\to j}\right)$$

$$\tilde{V}^{c} \xrightarrow{\mathcal{L}^{0}} \tilde{V}^{c\to 1} \xrightarrow{\mathcal{L}^{0\to 2}} \tilde{V}^{0\to 2} \to \cdots$$

Longest bars

$$\tilde{V}^{c} \to \tilde{V}^{c\to 1} \to \cdots \to \tilde{V}^{c\to n}$$

$$\dim(\tilde{V}^{c\to n}) = m_{c,n}$$

basis

$w_1^{c\to n}$

$w_2^{c\to n}$

$\vdots$

$w_{m_{c,n}}^{c\to n}$

so:

Observe $\bar{V}^{0} \xrightarrow{\mathcal{L}^{0} \cong \mathcal{L}^{0\to1}} \underbrace{\bar{V}^{0\to1}}_{\mathcal{L}^{0}(\bar{V}^{0})}$

$\mathcal{L}^{0\to1}$ surjective

$$\bar{V}^{0\to1} \longrightarrow \bar{V}^{0\to2} = \mathcal{L}^{1}(\bar{V}^{0\to1})$$

restriction of $\mathcal{L}^{1}$ to $\bar{V}^{0,1}$

So

$$\bar{V}^{0} \longrightarrow \bar{V}^{0\to1} \longrightarrow \bar{V}^{0\to2} \longrightarrow \cdots$$

all are surjective

Remark

$\bar{V}^d \quad \bar{V}^1 \quad \bar{V}^2 \quad \bar{V}^3 \quad \bar{V}^4$

no! $(c,3)$-bars

no! $(c,1)$-bars

$$\bar{V}^c \xrightarrow{\mathcal{L}^{c\to n}} \bar{V}^{c\to n}$$

surjective

Longest bars

$$V^{0} \to V^{0\to 1} \to \cdots \to V^{0\to n}$$

$$\dim(V^{0\to n}) = m_{0,n}$$

basis

$v_1^{0\to n} \to \cdots \to w_1^{0\to n}$

$v_2^{0\to n} \to \cdots \to w_2^{0\to n}$

$\vdots$

$w_{m_{0,n}}^{0\to n}$

so:

vectors in V^{0}

These are our $(0,n)$-bars

Observation $\qquad 0 \leq j \leq n$

$$\bar{V}^{0 \to j}$$

$$\left.\begin{array}{l} \mathcal{L}^{0\to j}\left(v_1^{0\to n}\right) \\ \mathcal{L}^{0\to j}\left(v_2^{0\to n}\right) \\ \vdots \\ \mathcal{L}^{0\to j}\left(v_{m_{0,n}}^{0\to n}\right) \end{array}\right\} \text{this are linearly independent}$$

Why?

Lemma: $M : \vec{V} \to \vec{W}$ and

$$\left.\begin{matrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{matrix}\right\} \text{linearly independent}$$

then if $v_i \in \vec{V}$ s.t. $M(v_i) = w_i$

for $1 \le i \le k$, then

$v_1, \ldots, v_k$ are lin indep.

Pf: Say $\sum_{i=1}^{k} \alpha_i v_i = 0$

then apply M

$$\sum_{i=1}^{k} \alpha_i w_i = 0.$$

$(0, n-1)$ - bar

$\bar V^{0}$ $\quad$ $\bar V^{0\to 1}$

$$v \longmapsto \mathcal{L}^{0\to 1} v \longmapsto \cdots \longmapsto \mathcal{L}^{0\to n-1} v \longmapsto 0$$

$\neq 0$

$$\bar V^{0} \to \bar V^{0\to 1} \to \cdots \to V^{0\to n-1} \to V^{0\to n}$$

$(0, n-1)$ bars stop here

So $\mathcal{L}^{0\to n-1} v \in \ker \mathcal{L}^{n-1}$

$$K_{0,n-1} = \ker(\mathcal{L}^{n-1}) \cap \bar V^{0\to n-1}$$

$$\tilde{V}^{0} \hookrightarrow \tilde{V}^{0\to 1} \hookrightarrow \cdots \hookrightarrow \tilde{V}^{0\to n-1} \longrightarrow \tilde{V}^{0\to n}$$

$$\{v_i^{0\to n}\} \xrightarrow{(0,n]\text{-bws}} \{w_i^{0\to n}\}$$

$$\{v_i^{0\to n-1}\} \xrightarrow{(0,n-1]\text{-bws}} w_i^{0\to n-1} \longmapsto 0$$

$\{w_i^{0\to n-1}\}$ basis for

$$K_{0,n-1} = \ker(\mathcal{L}^{n-1}) \cap \tilde{V}^{0\to n-1}$$

$$= \{v \in \tilde{V}^{0\to n-1} \mid \mathcal{L}^{n-1} v = 0\}$$

Similarly for all $(0,q]$-bws, $q = n-2,$ $n-3, \dots, 0$

Similarly: $(0,q)$-bars

$$V^0 \to \cdots \to V^{0\to q} \to V^{0\to q+1}$$

$$v \mapsto \mathcal{L}^{0\to 1} v \mapsto \cdots \mathcal{L}^{0\to q} v \mapsto 0$$

$$K_{0,q} = V^{0\to q} \cap \ker(\mathcal{L}^q)$$

$$= \{ w \in V^{0\to q} \mid \mathcal{L}^q w = 0 \}$$

basis:

$$\begin{array}{lllll} & & & & K_{0,q} \\ v_1^{0\to q} \mapsto & \mapsto & \cdots & \mapsto & w_1^{0\to q} \\ v_2^{0\to q} \mapsto & \cdots & & & w_2^{0\to q} \\ \vdots & & & & \vdots \end{array} \Bigg\} \begin{array}{c}(0,q) \\ \text{bars}\end{array}$$

(c,n)-bars:

$$m_{c,n} = \dim\left(\bar{V}^{c \to n}\right)$$

$(c,n-1)$-bars

$$\dim(K_{c,n-1})$$

$$\bar{V}^{c \to n-1} \xrightarrow[\text{surjective}]{} \bar{V}^{c,n}$$

$$\dim(K_{c,n-1}) = \boxed{\dim\left(\bar{V}^{c \to n-1}\right) - \dim\left(\bar{V}^{c \to n}\right)}$$