CPSC 531F April 4, 2025

Please email me homework by April 23.

Email during April for office hours.

There is one barcode exercise:

$H_0(K^i)$

$$V^0 \xrightarrow{\mathcal{L}^0} V^1 \xrightarrow{\mathcal{L}^1} V^2 \to \cdots \to V^n$$

Phase 0: Build all $(0,q)$-bars

① Their direct sum should equal

$$V^0 \to V^{0\to 1} \to V^{0\to 2} \to \cdots \to V^{0\to n}$$

surjective

$i \le j$: $V^{i\to j} = \mathrm{Image}(\mathcal{L}^{i\to j})$

$$\mathcal{L}^{i\to j} : V^i \to V^j$$

$$\parallel$$

has to be

$$\mathcal{L}^{j-1} \cdots \mathcal{L}^{i+1} \mathcal{L}^{i}$$

$$\mathcal{L}^{0\to q} \stackrel{\text{def}}{=} \mathcal{L}^{0\to q}(V^0)$$

$$= \mathcal{L}^{q-1}\left(\mathcal{L}^{q-2}\cdots\mathcal{L}^{0}(V^0)\right)$$

$$= \mathcal{L}^{q}\Big(\underbrace{\mathcal{L}^{0\to q-1}(V^0)}_{\overline{V}^{0\to q-1}}\Big)$$

$(0,q)$-bar:

has to lie in $\ker(\mathcal{L}^q)$

$$v \in \overline{V}^0 \mapsto \mathcal{L}^0 v \mapsto \mathcal{L}^{0\to 2} v$$

$$\mapsto \cdots \mapsto \mathcal{L}^{0\to q} v \mapsto 0$$

$$\begin{array}{cccccc} \bullet & - & \bullet & \cdots & \bullet & \to 0 \\ V^0 & & V^1 & & V^q & \end{array}$$

$$K_{0,q} \overset{\text{def}}{=} \ker(\mathcal{L}^q) \cap \bar{V}^{0\to q}$$

recipe: choose a basis

$$\{\omega_i^{0,q}\}_{i=1}^{\dim(K_{0,q})} \quad \text{for} \quad K_{0,q}$$

$$\underbrace{V^0 + V^{0\to 1} \to \cdots}_{\text{surjective}} \quad V^{0\to q}$$

$$\cup$$

$$\{\omega_i^{0,q}\}_{i=1} \quad K_{0,q}$$

$v_i^{0,q}$ s.t. $\mathcal{L}^{0\to q}(v_i^{0,q}) = \omega_i^{0,q}$

in

$\bar{V}^0$

$$(*)\quad V^{c} \to V^{c+1} \to \cdots \to V^{c+n-1} \to V^{c+n}$$

$$\begin{array}{ccccc} \cup & \cup & & & \\ K_{c,0} & K_{c,1} & \cdots & K_{c,n-1} & \end{array}$$

basis

$\{v_i^{c,n}\}_i \dashrightarrow \{w_i^{c,n}\}_i$

(c, n) bars

basis

$\{v_i^{c,n-1}\}_i \dashrightarrow \{w_i^{c,n-1}\}_i$

(c, n-1) bars

$\dashrightarrow \{\ \}$

...

The sum of these bars really gives (*)

Augmented:

$$\tilde{V}^0 \xrightarrow{\mathcal{L}^0} \tilde{V}^1 \xrightarrow{\mathcal{L}^1} \cdots \xrightarrow{\mathcal{L}^{n-1}} \tilde{V}^n \xrightarrow{\mathcal{L}^n} 0$$

$$0 = \tilde{V}^{n+1}$$

$$K_{i,j} = \ker(\mathcal{L}^j) \cap \tilde{V}^{i \to j}$$

$$K_{0,n} = \underbrace{\ker(\mathcal{L}^n)}_{\text{everything}} \cap \tilde{V}^{0 \to n}$$

$$\text{everything} \cap (\qquad)$$

$$= \tilde{V}^{0 \to n}$$

Conceptual Lemma: Linear map

$f: \bar{U} \to \bar{W}$ and

$u_1, \dots, u_r \in \bar{U}$ and

$w_i = f(u_i)$ if $w_1, \dots, w_r$

are linearly independent, then so

are $u_1, \dots, u_r$.

$$V^{0\to j} \longrightarrow \cdots \quad V^{0\to n}$$

$$\{v_i^{0,n}\} \mapsto \mathcal{L}^{0\to j}\{v_i^{0,n}\} \mapsto \cdots \quad \{w_i^{0,n}\}$$

lin

independent

Now: (c,n)-bars $\quad (c,n-1)$ bars

$$\{v_i^{c,n}\}_i \hookrightarrow \hookrightarrow \cdots \longrightarrow \{w_i^{c,n}\}$$

$$\{v_{i'}^{c,n-1}\}_{i'} \hookrightarrow \cdots \{w_i^{c,n-1}\} \longrightarrow 0$$

$$\bar{V}^c \qquad\qquad V^{c\to n-1} \longrightarrow \bar{V}^{c\to n}$$

Claim: for any j, $0 \le j \le n-1$

$$\bar{V}^j$$

$$\left.\begin{array}{l} \{\mathcal{L}^{c\to j}\, v_i^{c,n}\} \\ \{\mathcal{L}^{c\to j}\, v_{i'}^{c,n-1}\} \end{array}\right\} \text{ are linearly independent}$$

Note: it suffices to look at $j = n-1$

(c,n) bars: $\left\{ \mathcal{L}^{c \to n} V_i^{c,n} \right\}_i$

$(c,n-1)$ bars: $\left\{ \mathcal{L}^{c \to n-1} V_{i'}^{c,n-1} \right\}_{i'}$

$$\nabla^{c \to n-1}$$

Say that

$$\left(\sum_i \alpha_i \mathcal{L}^{c \to n-1} V_i^{c,n} + \sum_{i'} \beta_{i'} \mathcal{L}^{c \to n-1} V_{i'}^{c,n-1} = 0 \right) \quad (*)$$

First: claim: all α_i's $=0$:

Apply $\mathcal{L}^{n-1}$ to $(*)$

$$\sum_i \alpha_i \overbrace{\mathcal{L}^{c\to n} v_i^{c,n}}^{w_i^{c,n} \,\leftarrow\, \text{basis}} + \sum_{i'} \beta_{i'} \underbrace{\mathcal{L}^{c\to n} v_{i'}^{c,n-1}}_{=0} = 0$$

$\Downarrow$

$$\sum_i \alpha_i w_i^{c,n} = 0$$

So α_i's $= 0$.

$$\cancel{\sum_i \alpha_i \mathcal{L}^{c\to n-1} V_i^{c,n}}$$

$$+ \sum_{i'} \beta_{i'} \underbrace{\mathcal{L}^{c\to n-1} V_{i'}^{c,n-1}}_{\{w_{i'}^{c,n-1}\}} = 0$$

basis for $K_{c,n-1} \subset V^{c\to n-1}$

linearly indep

$$\Rightarrow \text{all } \beta_{i'} = 0.$$

Hence $\mathcal{L}^{c\to n-1}$ $\overbrace{\{V_i^{c,n}\}_i \cup \{V_{i'}^{c,n-1}\}_{i'}}^{\mathcal{B}}$

$$\underbrace{\dim V^{c,n-1}}_{m_{c,n-1}} = ?$$

$$V^{c,n-1} \xrightarrow{\text{surj}} V^{c,n}$$

$$\dim m_{c,n-1} \qquad\qquad \dim m_{c,n}$$

$$|B| = m_{c,n} + \underbrace{\dim(K_{c,n-1})}_{m_{c,n-1} - m_{c,n}}$$

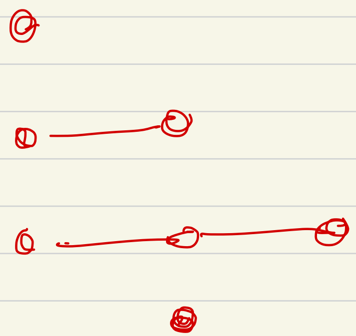
$$= m_{c,n-1}$$

Canonical ?

A

B

C

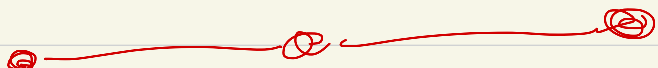


$$\overline{V}^{0} \to \overline{V}^{1} \to \overline{V}^{2}$$

$$\overline{V}^{0} \quad \overline{V}^{1} \quad \overline{V}^{2}$$

$$\mathbb{R}\{A, B, C\} \to \mathbb{R}\{B, C\} \to \mathbb{R}\{C\}$$

$17C - 3A + B$



17C + B   17C

(0,0)-bars:

$$K_{0,0} = \ker(\mathcal{L}^0) \cap \underbrace{\bar{V}^{0,0}}_{=\bar{V}^0}$$

$$= \ker(\mathcal{L}^0)$$

$$\bar{V}^0 \xrightarrow{\mathcal{L}^0} \bar{V}^{0,1} \to \dots$$

(0,0)-bars are canonical

$$\bar{V}^0 / K_{0,0} \xrightarrow{\sim} \bar{V}^{0,1}$$ for (0,1) bars

$<$ dim

$m_{0,1}$ $m_{0,1}$

$(c,1)$-bws

$$\sigma^{c,1}$$

$$\cup$$

$$K_{c,1} \xrightarrow{\mathcal{L}'} 0$$

$$\uparrow$$

choose

$$V^0 / K_{c,0} \overset{\sim}{\dashrightarrow} \{\omega_i^{c,1}\}_i$$

canonical

$$\omega_i^{c,1} + K_{c,0} \longmapsto \omega_i^{c,1}$$

Idea!

$$V^0 \longrightarrow V^{0\to 1} \longrightarrow V^{0\to 2} \longrightarrow \cdots$$

all the $(0,q)$-bars, $q \geq 0$

Now find $(1,q)$-bars, $q \geq 1$

$$V^{0\to 1} \qquad V^{0,2}$$

$$\cap$$

$$V^0 \qquad V^1 \longrightarrow V^{1\to 2} \longrightarrow \cdots$$

More generally

$(p,q) \quad q \geq p, \quad 0 \leq p \leq s-1$

$$\tilde{V}^{0} \to \tilde{V}^{1} \to \dots \to \tilde{V}^{s-1} \longrightarrow \tilde{V}^{s-1+s} \to \dots$$

leave this?

$$\dots \quad V^{s-2} \quad V^{s-1} \qquad \tilde{V}^{s} \to \dots$$