CPSC 531F April 7, 2025

Cancel office hours today.

Thursday: email me if you want to attend.

[Next few days will likely be noisey - location of hours to be moved.]

Today:

(1) Finish barcode decomposition

(2) Stare at a few examples in

textbook:

(2020)

TDA in Genomics & Evolution

- Nice pictures, nice algorithms

Inductively:

We've done $(0,q)$-bar, $0 \leq q \leq n$

whose sum is

$$V^{0} \to V^{0\to 1} \to V^{0\to 2} \to \cdots \to V^{0\to n}$$

next step: build $(1,q)$-bars for

$1 \leq q \leq n$

$$\bullet \longrightarrow \circ \longrightarrow \circ \to 0$$

$$\bar{V}^{1}$$

Summing $(0,q)$-bars, $(1,q')$-bars:

$\bar{V}^{0} \to \bar{V}^{1} \to$ part of $\bar{V}^{2}$ containing images of $\bar{V}^{0}, \bar{V}^{1}$ in $\bar{V}^{2}$

enough to say $\left\{ \begin{array}{l} \text{Image } \mathcal{L}^{0\to 2} \text{ or } \mathcal{L}^{0\to 2}\bar{V}^0 \\ \oplus\ \text{Image } \mathcal{L}^{1\to 2} \text{ or } \mathcal{L}^{1\to 2}\bar{V}^1 \end{array} \right.$

$$\text{Image } \mathcal{L}^{1\to 2} = \mathcal{L}'(\bar{V}^1) = \bar{V}^{1\to 2}$$

So need: enough $(1,q)$-bars

to add to

end of phase 0 $\left\{ \bar{V}^0 \longrightarrow \bar{V}^{0\to 1} \longrightarrow \bar{V}^{0\to 2} \longrightarrow \cdots \right.$

to get $\quad\bigcap \qquad \bigcap \qquad \cdots$

$$\bar{V}^0 \longrightarrow \bar{V}^1 \longrightarrow \bar{V}^{1\to 2} \longrightarrow \bar{V}^{1\to 3} \longrightarrow \cdots$$

(end of phase)

METHOD 1

Could think "more canonically"

as

$$\underset{\displaystyle 0}{\overset{\displaystyle \wr\!\shortparallel}{\tilde{V}^0 / \tilde{V}^0}} \longrightarrow \underbrace{\tilde{V}^1 / \tilde{V}^{0\to 1} \longrightarrow \tilde{V}^2 / \tilde{V}^{0\to 2} \longrightarrow \cdots}_{\text{induction} \ldots}$$

then "lift" every bar to

$$\tilde{V}^1 \longrightarrow \tilde{V}^{1\to 2} \longrightarrow \tilde{V}^{1\to 3} \longrightarrow \cdots$$

METHOD 2:

Perform something similar for the $(1,q)$-bars that we did for $(0,q)$-bars

Then we'll have

$$\bar{V}^{0} \to \bar{V}^{1} \to \bar{V}^{1\to 2} \to \bar{V}^{1\to 3} \to \cdots$$

next phase: add $(0,2)$-bars:

end of phase 1

$$\bar{V}^{0} \to \bar{V}^{1} \to \bar{V}^{1\to 2} \to \bar{V}^{1\to 3} \to \cdots$$

Phase 2
add $(2,q)$-bars
to get

$$\bar{V}^{0} \to \bar{V}^{1} \to \bar{V}^{2} \to \bar{V}^{2\to 3} \to \cdots$$

Phase 0 to Phase 1

We want: $(1,n)$-bars

$$\bar{V}^0 \to \bar{V}^{0\to 1} \to \bar{V}^{0\to 2} \to \cdots \to \bar{V}^{0\to n} \to 0$$

$$\bar{V}^{0\to n+1} = 0 \quad (\bar{V}^{n+1} = 0)$$

$$\circ \to \circ \to \cdots \to \circ$$

$$\bar{V}^1 \qquad\qquad \bar{V}^n$$

$$\bar{V}^1 \to \bar{V}^{1\to 2} \to \cdots \to \bar{V}^{1\to n} \to 0$$

We pick a basis for

$$\bar{V}^{1\to n} / \bar{V}^{0\to n}$$

Definition:

Given $\bar{U} < \bar{W}$ vector spaces:

a basis of $\bar{W}$ relative to $\bar{U}$:

$w_1, \ldots, w_r$, $r = \dim(\bar{W}) - \dim(\bar{U})$

s.t.

① Every vector in $\bar{W}$ can be written uniquely as a linear combination of $w_1, \ldots, w_r$ plus some vector in $\bar{U}$

② $w_1 + \bar{U}, w_2 + \bar{U}, \ldots, w_r + \bar{U} \in \bar{W}/\bar{U}$

these r-vectors in quotient $\bar{W}/\bar{U}$ are a basis

(3) If $u_1, \ldots, u_m$ are any basis of $\overline{U}$, then

$$u_1, \ldots, u_m, w_1, \ldots, w_r$$

are a basis of $\overline{W}$

(4) $\overline{W}$ is an internal direct sum, of direct sum of subspaces

$$\mathrm{Span}(w_1), \mathrm{Span}(w_2), \ldots, \mathrm{Span}(w_r), \overline{U}$$

(4') $\overline{W}$ is ~ ~ of

$$\mathrm{Span}(w_1, w_2, \ldots, w_r), \overline{U}$$

(4'') (5) (6) (7) $w_1, \ldots, w_r$ represent "lifting" a basis of $\overline{W}/U$

(8) (9) ~ ~

So pick a relative basis

$w_1^{1\to n}, \ldots, w_s^{1\to n}$ of $\bar{V}^{1\to n}$

relative to $\bar{V}^{0\to n}$

$V^{0\to n} \subset \bar{V}^{1\to n}$

$$\begin{array}{cccc} w_1^{1\to n}, & \ldots, & w_s^{1\to n} & \in \bar{V}^{1\to n} \\ \text{so} \uparrow \mathcal{L}^{1\to n} & & \uparrow \mathcal{L}^{1\to n} & \\ v_1^{1\to n}, & \ldots, & v_s^{1\to n} & \bar{V}^{1} \end{array}$$

So we have

$$V_1^{0\to n}, V_2^{0\to n}, \ldots, V_{m_{0,n}}^{0\to n}$$

$$m_{0,n} = \dim\left(\bar{V}^{0\to n}\right)$$

$$V_1^{1\to n}, \ldots, V_s^{1\to n} \text{ in } \bar{V}^1$$

So that

$$\left.\begin{array}{l} \left\{\mathcal{L}^{0\to n}\left(V_i^{0\to n}\right)\right\}_{i\in[m_{0,n}]} \cup \\ \left\{\mathcal{L}^{1\to n}\left(V_i^{1\to n}\right)\right\}_{i\in[s]} \end{array}\right\} \mathcal{B}$$

we get a basis for $\bar{V}^{1\to n}$

V^n

$\cup$

basis B $\rightarrow$ $V^{1\to n}$

$\cup$

$V^{0\to n}$

$\{V_i^{0\to n}\}$ $\{\sum^{0\to 1} V_i^{0\to n}\}$

$\{V_i^{1\to n}\}$

V^0 V^1 $\longrightarrow$ $\longrightarrow$ V^n

basis

Claim: any $1 \le j \le n$

$$\mathcal{L}^{0\to j}\{V_i^{0\to n}\}$$

$$\mathcal{L}^{1\to j}\{V_i^{1\to n}\}$$

are a basis for $\bar{V}^{1\to j}$.

These are linearly independent in $\bar{V}^{\to j}$

and claim:

$$\#\ \text{vectors} = \dim\left(\bar{V}^{1\to n}\right)$$

$\vdots$