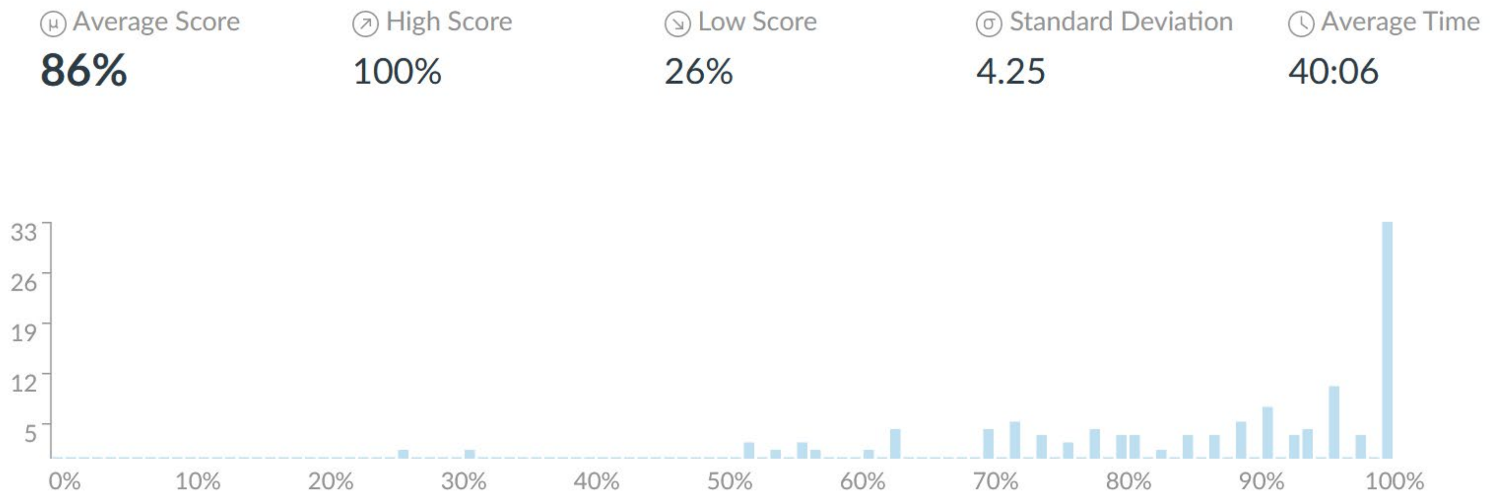
CPSC 340: Machine Learning and Data Mining

Multi-Class Linear Classifiers

Summer 2021

Admin



- Canvas grades released. Good job!
- Assignment 4 due Monday.
  - GO TO TUTORIALS. THEY'RE VERY HELPFUL.
- Assignment 5 out Friday.
- Final exam is Wednesday, June 23, 2021
  - Probably similar format as midterm.

Last Time: SVM and Logistic Regression

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\} + \frac{1}{2}\|w\|^2$$

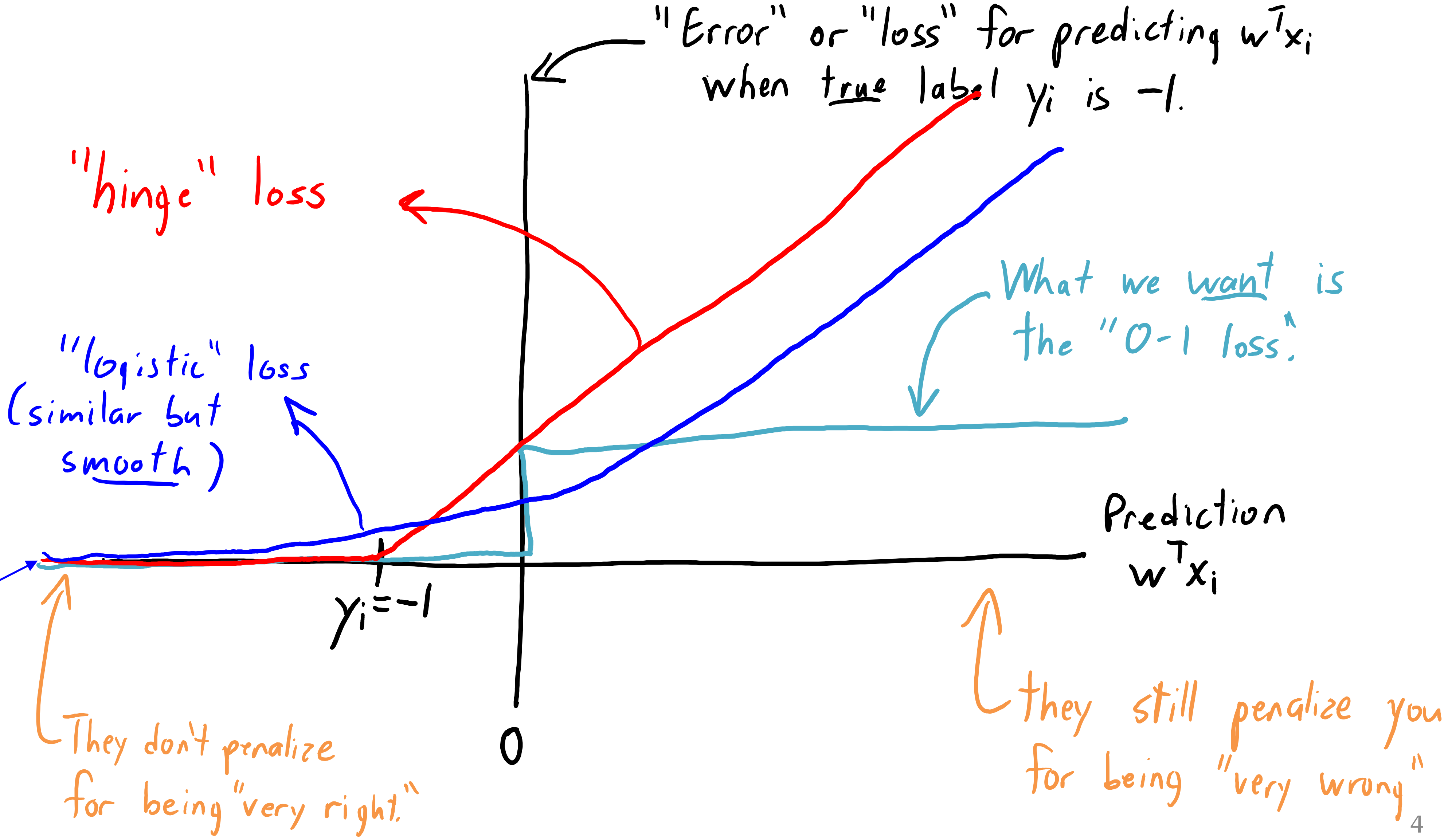
Hinge loss for support vector machine

$$f(w) = \sum_{i=1}^{n} \log(1 + \exp(-y_i w^T x_i))$$

Logistic loss for logistic regression

- We derived these losses step-by-step.
- We will do similar stuff today.

Last Time: SVM and Logistic Regression



In This Lecture

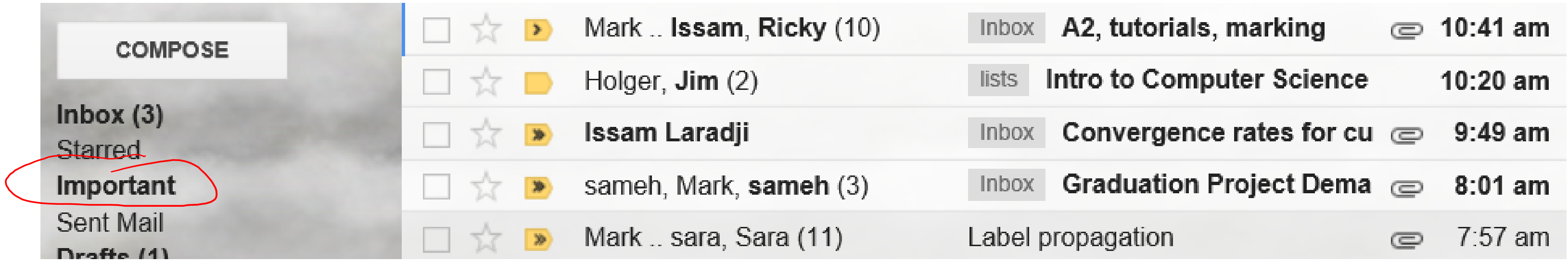
1. Linear Probabilistic Classifiers (10 minutes)
2. Multi-Class Classification Intro (10 minutes)
3. Multi-Class SVM (20 minutes)
4. Multi-Class Logistic Regression (15 minutes)

Coming Up Next

LINEAR PROBABILISTIC CLASSIFIERS

Previously: Identifying Important E-mails

- Recall problem of identifying 'important' e-mails:



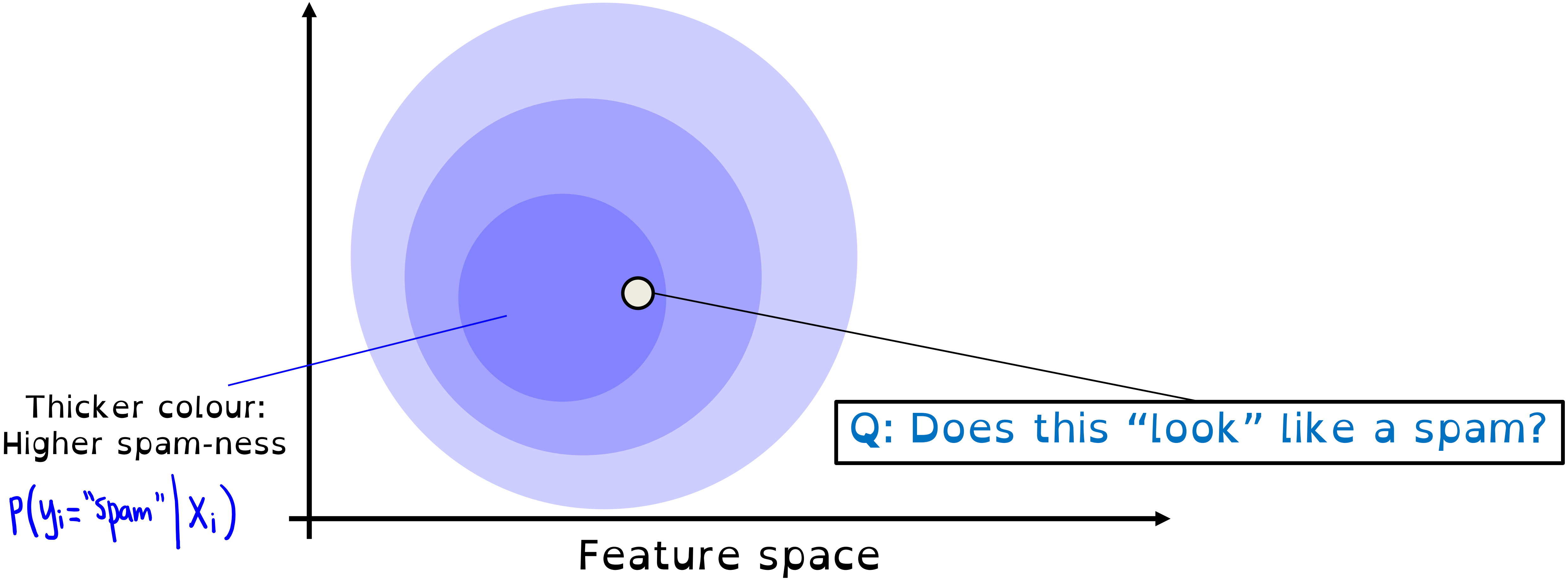
- We can do binary classification by taking sign of linear model:

$$\hat{y}_i = \text{sign}(w^T x_i)$$

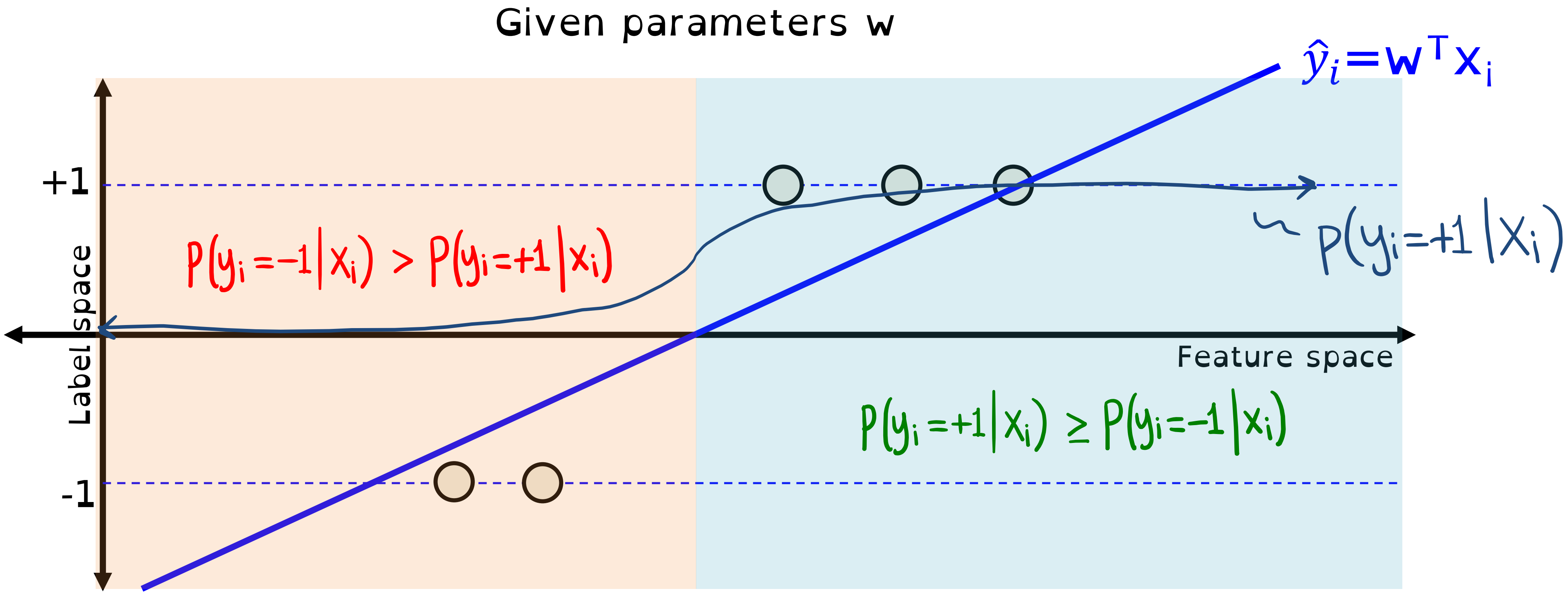
  - Convex loss functions (hinge/logistic loss) let us find an appropriate 'w'.

- But what if we want a probabilistic classifier?
  - Want a model of $p(y_i = \text{“important”} \mid x_i)$ for use in decision theory.

Recall: “Spam-ness”



Linear Prediction of "+1-ness"



Q: How should $p(y_i=+1 \mid x_i)$ behave?

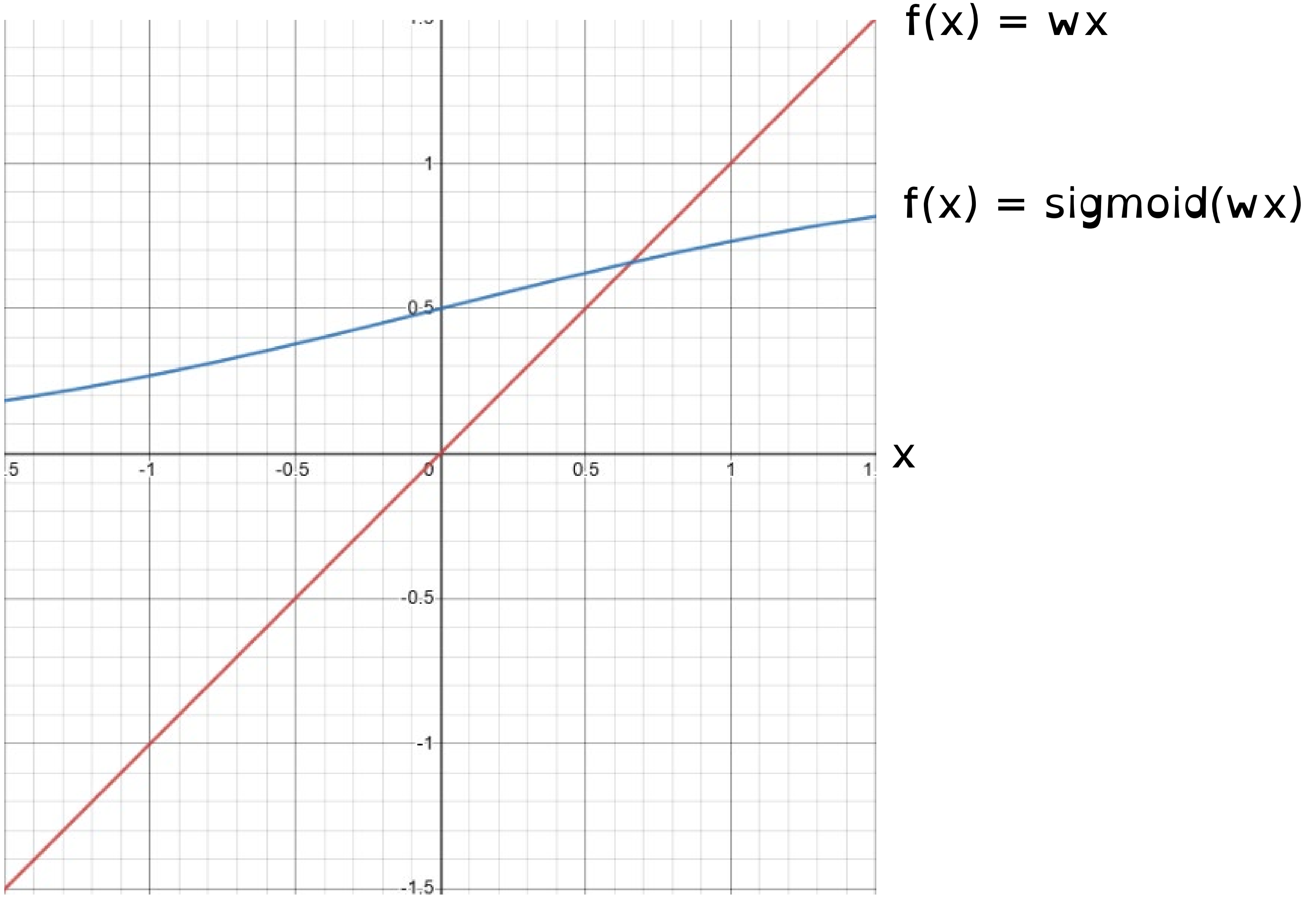
Sigmoid Function

Sigmoid: $\mathbb{R} \rightarrow (0, 1)$

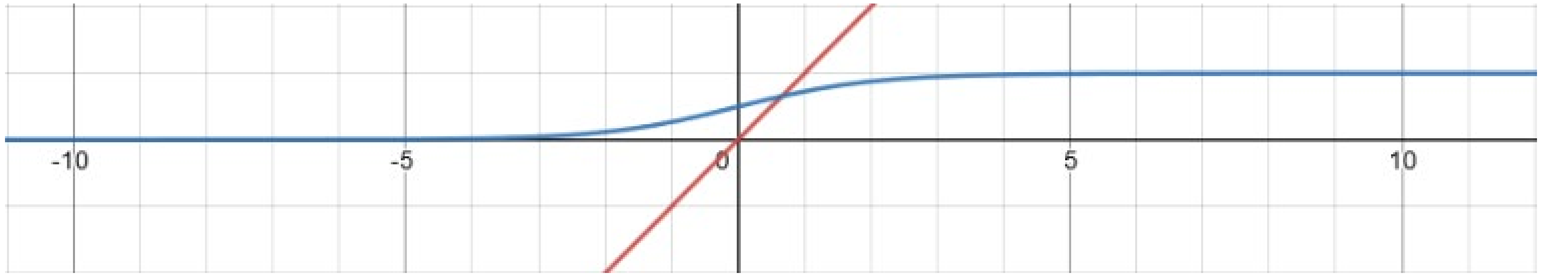
$$\text{Sigmoid}(z) = \frac{1}{1+\exp(-z)}$$

Q: What is sigmoid(z) when z is negative?

What is sigmoid(z) when z is positive?



+1-ness with Sigmoid



f(x) = sigmoid(wx)

- Idea: let's compute +1-ness with sigmoid.
- Given parameters **w**:
    1. Compute $z_i = \mathbf{w}^T x_i$
    2. Compute $p(y_i = +1 \mid \mathbf{w}, x_i) = \text{sigmoid}(z_i)$

$$P(y_i = +1 \mid w, \cdot) : \mathbb{R}^d \to (0, 1)$$

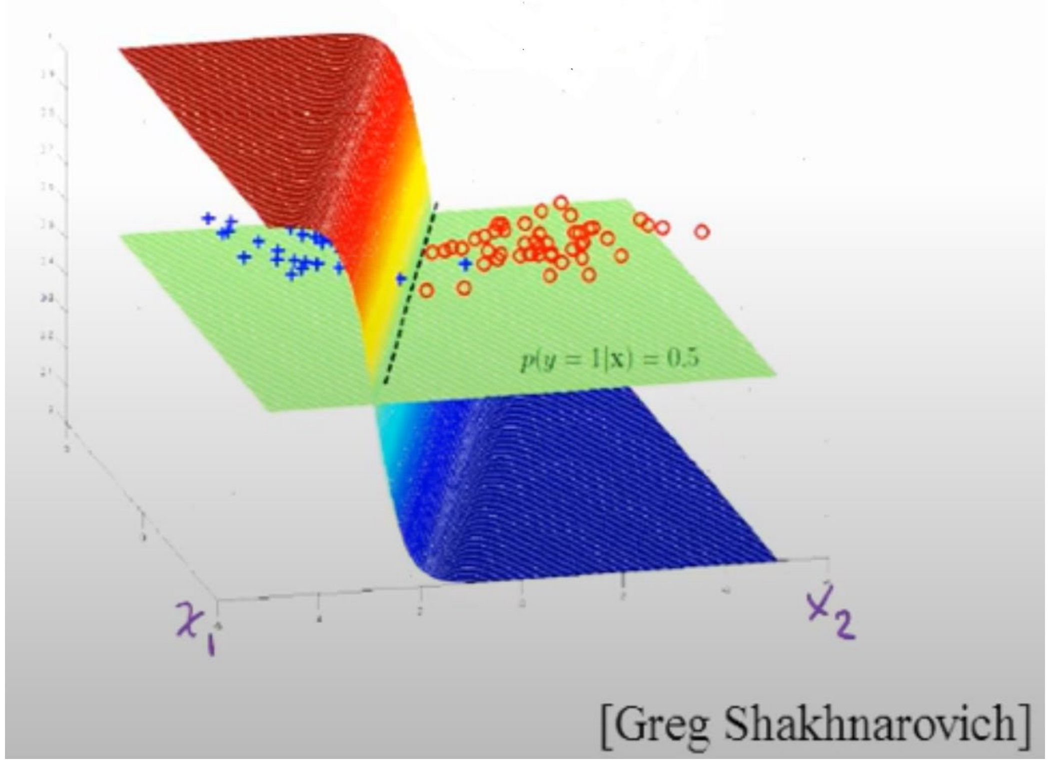
$$p(y_i = +1 \mid w, x_i) = \text{sigmoid}(w^T x_i)$$

Probabilities for Linear Classifiers using Sigmoid

- Using sigmoid function, we output probabilities for linear models using:

$$p(y_i = +1 \mid w, x_i) = \frac{1}{1 + \exp(-w^T x_i)}$$

- Visualization for 2 features:



What About "-1-ness"?

- Using sigmoid function, we output probabilities for linear models using:

$$p(y_i = +1 \mid w, x_i) = \frac{1}{1 + \exp(-w^T x_i)}$$

- By rules of probability:

$$p(y_i = -1 \mid w, x_i) = 1 - p(y_i = 1 \mid w, x_i)$$

$$= \frac{1}{1 + \exp(w^T x_i)} \quad \text{(with some effort)}$$

- We then use these for "probability that an email x_i is important".
- This may seem heuristic, but later we'll see that:
  - minimizing logistic loss does "maximum likelihood estimation" in this model.

Coming Up Next

MULTI-CLASS CLASSIFICATION INTRO

Multi-Class Linear Classification

- We've been considering linear models for binary classification:

$$X = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- E.g., is there a cat in this image or not?

Multi-Class Linear Classification

- Today we'll discuss linear models for multi-class classification:

$$X = \begin{bmatrix} \end{bmatrix} \qquad y = \begin{bmatrix} 27 \\ 16 \\ 8 \\ 7 \\ 21 \\ 5 \end{bmatrix}$$

- For example, classify image as "cat", "dog", or "person".
  - This was natural for methods of Part 1 (decision trees, naïve Bayes, KNN).
  - For linear models, we need some new notation.

Q: Can we use binary classifiers for multi-class?

"One vs All" Classification

- Training phase:
  - For each class 'c', train binary classifier to predict whether example is a 'c'.
    - For example, train a "cat detector", a "dog detector", and a "human detector".
    - If we have 'k' classes, this gives 'k' binary classifiers .

$$X = n\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}_{d} \qquad y = n\begin{bmatrix} \text{"cat"} \\ \text{"dog"} \\ \text{"human"} \\ \text{"cat"} \end{bmatrix}_{1} \longrightarrow y_{cat} = \begin{bmatrix} +1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$$

$$\underbrace{(X, y_{cat}) \rightarrow w_{cat}}_{\text{"cat detector"}} \qquad (X, y_{dog}) \rightarrow w_{dog} \qquad (X, y_{human}) \rightarrow w_{human}$$

"One vs All" Classification

$$W = \begin{array}{cc} & \\ k & \begin{bmatrix} \text{——} \; w_{cat} \; \text{——} \\ \text{——} \; w_{dog} \; \text{——} \\ \\ \text{——} \; w_{human} \; \text{——} \end{bmatrix} \\ & d \end{array}$$

classes

- Prediction phase:
  - Apply the 'k' binary classifiers to get a "score" for each class 'c'.
  - Predict the 'c' with the highest score.

features x_i

$$w_{cat}^T x_i = -0.1$$
$$w_{dog}^T x_i = -0.8$$
$$w_{human}^T x_i = 0.9$$

$$\left.\begin{array}{l} w_{cat}^T x_i = -0.1 \\ w_{dog}^T x_i = -0.8 \\ w_{human}^T x_i = 0.9 \end{array}\right\} \hat{y}_i = \text{human}$$

Multi-Class Linear Classification (MEMORIZE)

- Back to multi-class classification where we have 1 "correct" label:

$$X = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad y = \begin{bmatrix} 27 \\ 16 \\ 8 \\ 7 \\ 21 \\ 5 \end{bmatrix} \qquad W = \begin{bmatrix} \text{---} \; w_1^T \; \text{---} \\ \text{---} \; w_2^T \; \text{---} \\ \vdots \\ \text{---} \; w_k^T \; \text{---} \end{bmatrix}$$

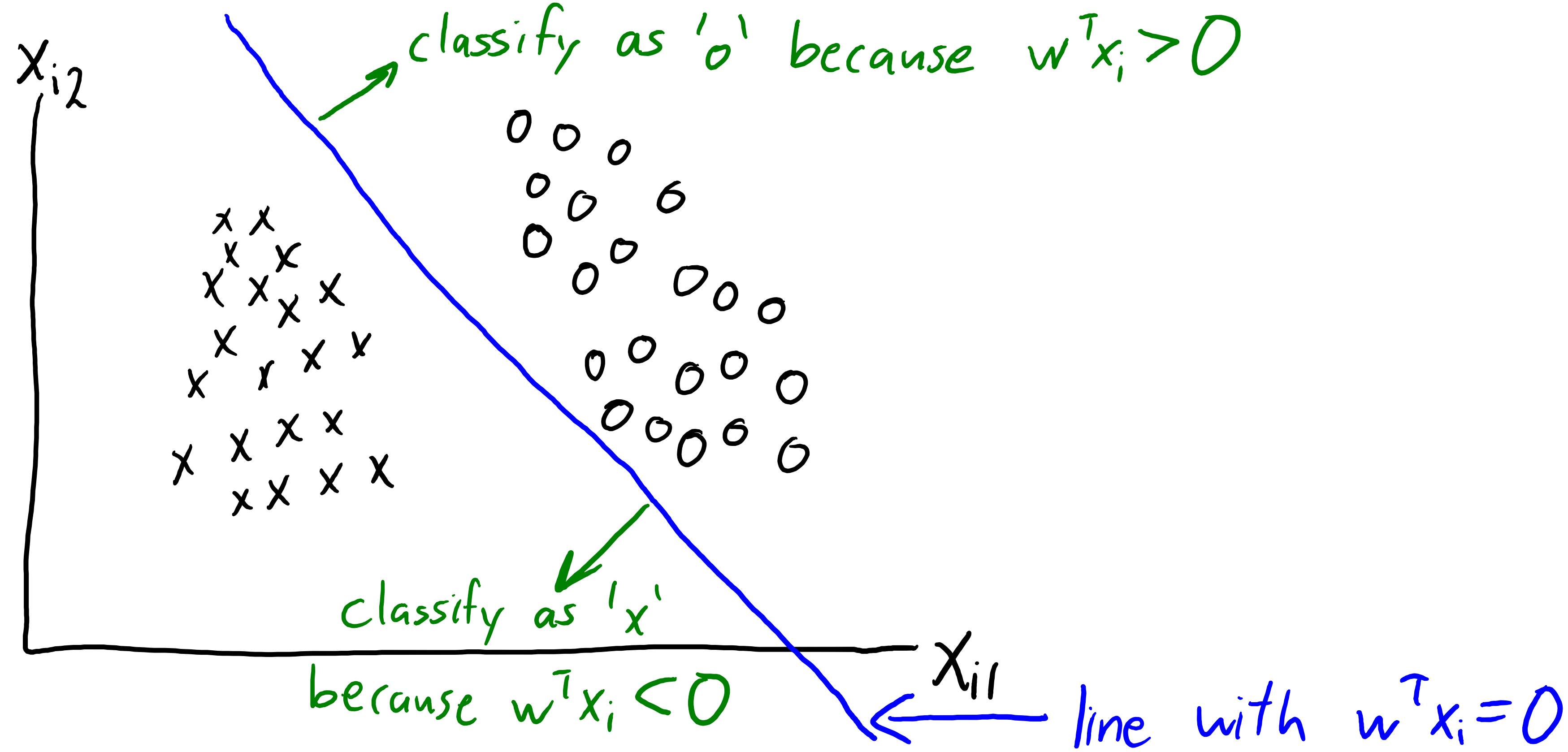
train 'k' classifiers

Predict by maximizing $w_c^T x_i$

- We'll use 'w_{y_i}' as classifier where $c=y_i$ (row of correct class label).
  - So if $y_i=2$ then $w_{y_i} = w_2$.

Shape of Decision Boundaries

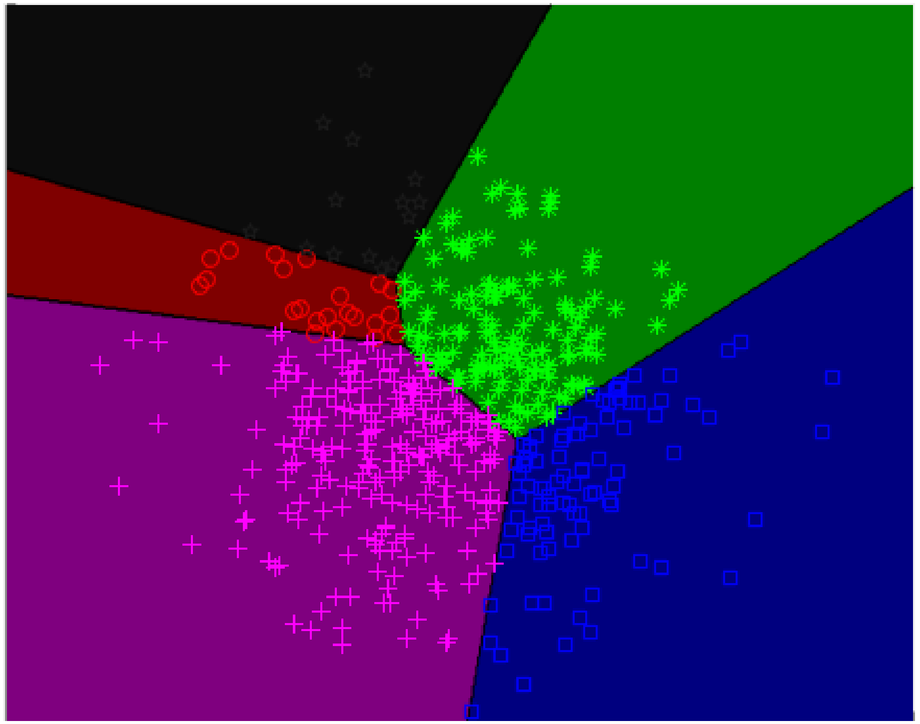
- Recall that a binary linear classifier splits space using a hyper-plane:



- Divides x_i space into 2 "half-spaces".

Shape of Decision Boundaries

- Multi-class linear classifier is intersection of these “half-spaces”:
  - This divides the space into convex regions (like k-means):



"Blue" region is region where we have:

$w_{blue}^T x_i \geq w_{green}^T x_i$

$w_{blue}^T x_i \geq w_{magenta}^T x_i$

$w_{blue}^T x_i \geq w_{red}^T x_i$

$w_{blue}^T x_i \geq w_{black}^T x_i$

  - Could be non-convex with change of basis.

Digression: Multi-Label Classification

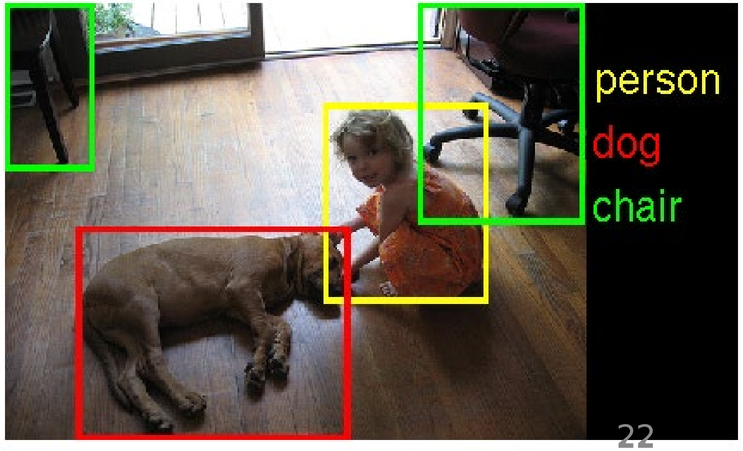
- A related problem is multi-label classification:
-

$$X = \underbrace{\left[\quad\quad\quad\quad\right]}_{d}\Big\}\, n$$

$$Y = \underbrace{\begin{array}{ccccc} \text{cat} & \text{dog} & \text{person} & \text{chair} & \text{mouse} \\ \end{array}}{} \underbrace{\begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}}_{k}\Big\}\, n$$

$$W = \underbrace{\begin{bmatrix} \text{---}\; w_1^T \;\text{---} \\ \text{---}\; w_2^T \;\text{---} \\ \vdots \\ \text{---}\; w_k^T \;\text{---} \end{bmatrix}}_{d}\Big\}\, k$$

- Which of the 'k' objects are in this image?
  - There may be more than one "correct" class label.
  - Here we can also fit 'k' binary classifiers.
    - But we would take all the $\text{sign}(w_c^T x_i)=+1$ as the labels.



http://image-net.org/challenges/LSVRC/2013/

"One vs All" Multi-Class Linear Classification

- Problem: We didn't train the w_c so that the largest $w_c^T x_i$ would be $w_{y_i}^T x_i$.
  - Each classifier is just trying to get the sign right.

features x_i

$w_1^T x_i = -5$ ("cat" score)

$w_2^T x_i = -0.1$ ("dog" score)

$w_3^T x_i = -0.2$ ("human" score)

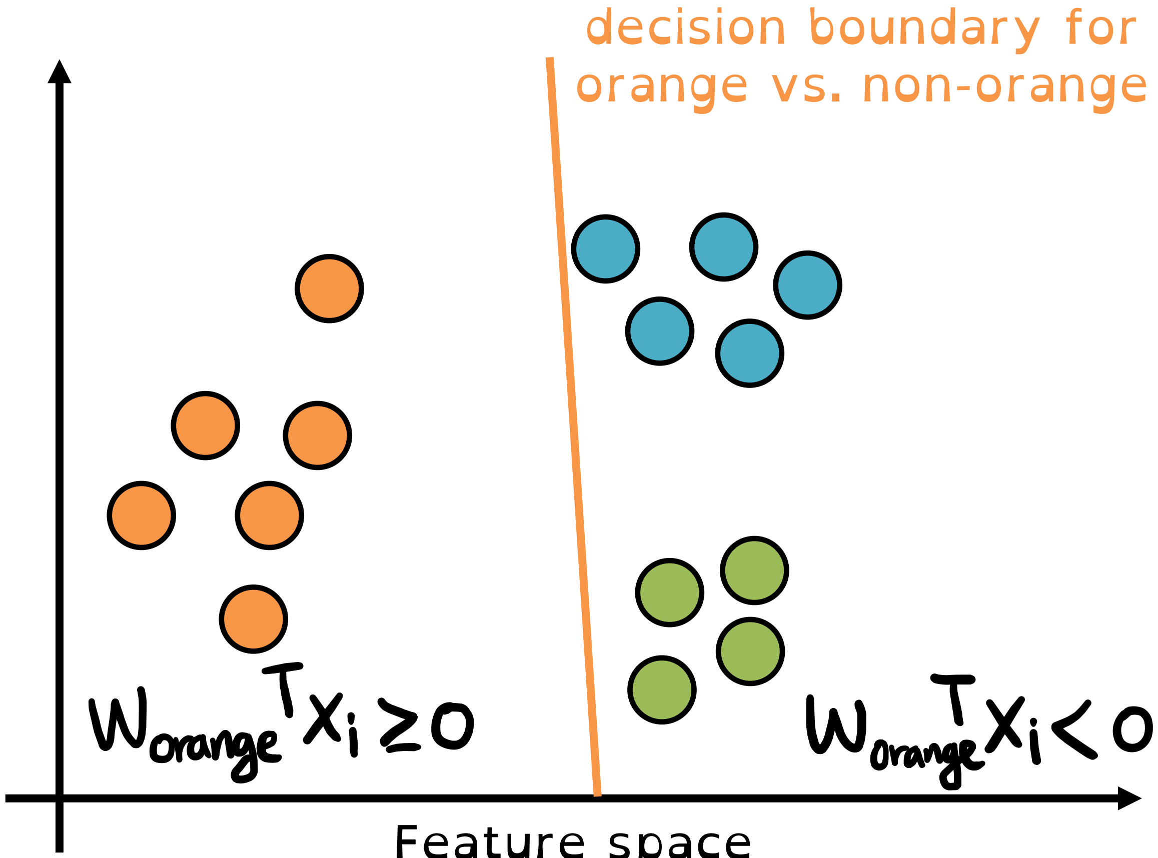
  - Here the classifier incorrectly predicts "dog".
    - "One vs All" doesn't try to put $w_2^T x_i$ and $w_3^T x_i$ on same scale for decisions like this.
    - We should try to make $w_3^T x_i$ positive and $w_2^T x_i$ negative relative to each other.
    - The multi-class hinge losses and the multi-class logistic loss do this.

https://laughingsquid.com/pug-mask-a-latex-mask-so-you-can-look-like-a-dog/

Coming Up Next

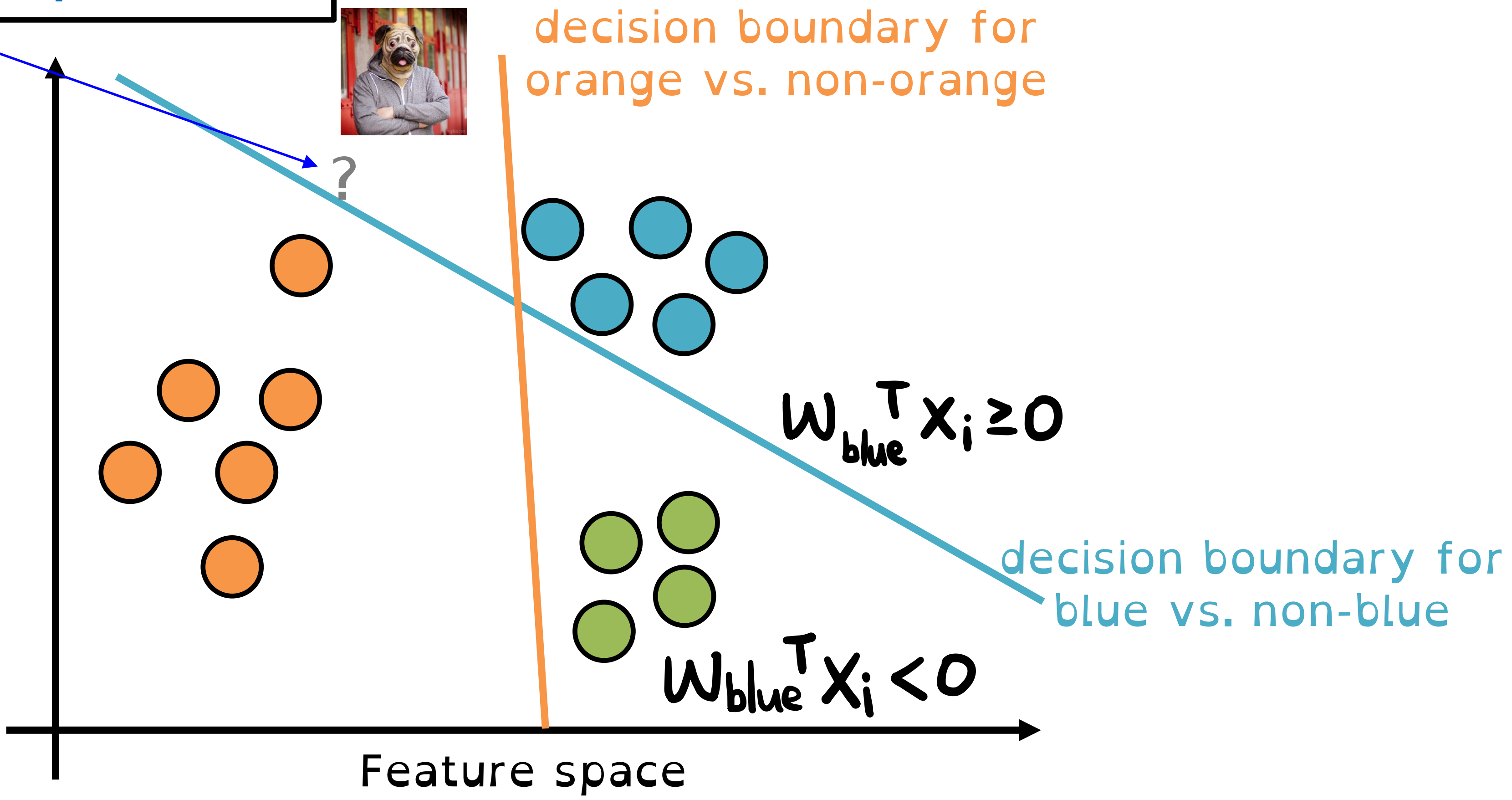
MULTI-CLASS SVM

Binary Classifiers are "Under-constrained"



Binary Classifiers are "Under-constrained"

Q: What's the prediction for this example?



What Do We Want for Multi-Class Classifiers?

- Idea: additional constraints on slopes
  - Will "force" classifier to find better boundaries

- Think of $w_{cat}^T x_i$, $w_{dog}^T x_i$, $w_{human}^T x_i$ as scores.
  - Previously, we only wanted $w_{cat}^T x_i > 0$ (underconstrained!)
  - New constraint: "cat" example x_i should have higher $w_{cat}^T x_i$ than $w_{dog}^T x_i$, $w_{human}^T x_i$
  - Now, we want: $w_{cat}^T x_i > w_{dog}^T x_i$ and $w_{cat}^T x_i > w_{human}^T x_i$

Q: How should we design the error here?

Multi-Class Loss Function

Now, we want: $w_{cat}^T x_i > w_{dog}^T x_i$ and $w_{cat}^T x_i > w_{human}^T x_i$

Let's count # times $W_{cat}^T X_i \leq W_{dog}^T X_i$ + # $W_{cat}^T X_i \leq W_{human}^T X_i$

[1] $$f_{cat}(W) = \sum_{i \in \text{cat examples}} I(W_{cat}^T X_i \leq W_{dog}^T X_i) + I(W_{cat}^T X_i \leq W_{human}^T X_i)$$

[2] $$= \sum_{i \in \text{cat examples}} I(0 \leq -W_{cat}^T X_i + W_{dog}^T X_i) + I(0 \leq -W_{cat}^T X_i + W_{human}^T X_i)$$

[3] $$\approx \sum_{i \in \text{cat examples}} \max\{0, -W_{cat}^T X_i + W_{dog}^T X_i\} + \max\{0, -W_{cat}^T X_i + W_{human}^T X_i\}$$

Multi-Class Loss Function

- Let's generalize this!

[4] $$f_{cat}(W) = \sum_{i \in \text{cat examples}} \max\{0, -W_{cat}^T x_i + W_{dog}^T x_i\} + \max\{0, -W_{cat}^T x_i + W_{human}^T x_i\}$$

[5] $$f_c(W) = \sum_{i \in \text{"c" examples}} \sum_{c' \neq c} \max\{0, -W_c^T x_i + W_{c'} x_i\}$$

[6] $$f(W) = \sum_{c=1}^{k} f_c(W) = \sum_{c=1}^{k} \sum_{i \in \text{"c" examples}} \sum_{c' \neq c} \max\{0, -W_c^T x_i + W_{c'} x_i\}$$

[7] $$= \sum_{i=1}^{n} \sum_{c' \neq y_i} \max\{0, -W_{y_i}^T x_i + W_{c'} x_i\}$$

Multi-Class Loss Function

$$f(W) = \sum_{i=1}^{n} \sum_{c' \neq y_i} \max\{0, -W_{y_i}^T x_i + W_{c'} x_i\}$$

$$X = n\begin{bmatrix} \\ \\ \text{---}\ x_i\ \text{---} \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow i \end{array} \qquad y = n\begin{bmatrix} \text{"cat"} \\ \text{"dog"} \\ \text{"human"} \\ \text{"cat"} \end{bmatrix}\begin{array}{l} \\ \\ \\ \leftarrow i \end{array} \qquad W = \begin{array}{r} W_{y_i} \rightarrow \\ \\ k \\ \\ \end{array}\begin{bmatrix} \text{---}\ W_{cat}\ \text{---} \\ \text{---}\ W_{dog}\ \text{---} \\ \\ \text{---}\ W_{human}\ \text{---} \end{bmatrix}$$

$$\begin{array}{ccc} \quad d \quad & \quad 1 \quad & \quad d \quad \end{array}$$

$$= \max\{0, -W_{cat}^T x_i + W_{dog}^T x_i\} + \max\{0, -W_{cat}^T x_i + W_{human}^T x_i\}$$

Multi-Class Hinge Loss

$$f(W) = \sum_{i=1}^{n} \sum_{c' \neq y_i} \max\{0, -w_{y_i}^T x_i + w_{c'} x_i\}$$

- This function is degenerate: $f(0) =$ ___.
- As with binary SVM, we introduce an offset

$$f(W) = \sum_{i=1}^{n} \sum_{c' \neq y_i} \max\{0, 1 - w_{y_i}^T x_i + w_{c'} x_i\}$$

"sum"-rule multi-class hinge loss

$$f(W) = \sum_{i=1}^{n} \max_{c' \neq y_i}\{\max\{0, 1 - w_{y_i}^T x_i + w_{c'} x_i\}\}$$

"max"-rule multi-class hinge loss

Multi-Class SVMs

- Idea: for a cat example, we want: $w_{cat}^T x_i > w_{dog}^T x_i$ and $w_{cat}^T x_i > w_{human}^T x_i$

$$f(W) = \sum_{i=1}^{n} \sum_{c' \neq y_i} \max\{0, 1 - w_{y_i}^T x_i + w_{c'} x_i\}$$

$$f(W) = \sum_{i=1}^{n} \max_{c' \neq y_i} \{\max\{0, 1 - w_{y_i}^T x_i + w_{c'} x_i\}\}$$

- For each training example 'i':
  - "Sum" rule penalizes for each 'c' that violates the constraint.
  - "Max" rule penalizes for one 'c' that violates the constraint the most.
    - "Sum" gives a penalty of 'k-1' for W=0, "max" gives a penalty of '1'.
- If we add L2-regularization, both are called multi-class SVMs:
  - "Max" rule is more popular, "sum" rule usually works better.
  - Both are convex upper bounds on the 0-1 loss.

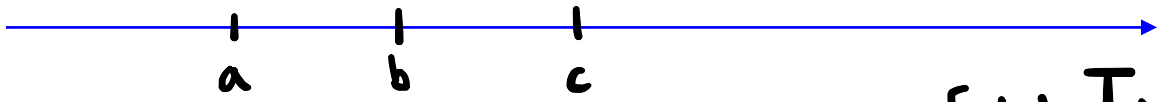
Coming Up Next

MULTI-CLASS LOGISTIC REGRESSION

Multi-Class Logistic Regression

- Idea: for a cat example, we want: $w_{cat}^T x_i > w_{dog}^T x_i$ and $w_{cat}^T x_i > w_{human}^T x_i$
  - In other words: **we want** $w_{cat}^T x_i$ **to be max** $w_c^T x_i$

$c > a \quad c > b \quad c = \max\{a, b, c\}$



$$\left.\begin{array}{l} W_{cat}^T X_i > W_{dog}^T X_i \\ W_{cat}^T X_i > W_{human}^T X_i \end{array}\right\} \Leftrightarrow W_{cat}^T X_i = \max \left\{ \begin{array}{l} W_{cat}^T X_i \\ W_{dog}^T X_i \\ W_{human}^T X_i \end{array} \right\}$$

Count #times $W_{cat}^T X_i < \max \left\{ \begin{array}{l} W_{cat}^T X_i \\ W_{dog}^T X_i \\ W_{human}^T X_i \end{array} \right\}$ or $0 < -W_{cat}^T X_i + \max \left\{ \begin{array}{l} W_{cat}^T X_i \\ W_{dog}^T X_i \\ W_{human}^T X_i \end{array} \right\}$

$\boxed{\text{Q: What happens when } W=0\text{?}}$

Multi-Class Logistic Regression

$$-W_{cat}^T x_i + \max\begin{Bmatrix} w_{cat}^T x_i \\ w_{dog}^T x_i \\ w_{human}^T x_i \end{Bmatrix} \approx -W_{cat}^T x_i + \log\begin{pmatrix} \exp(w_{cat}^T x_i) \\ + \\ \exp(w_{dog}^T x_i) \\ + \\ \exp(w_{human}^T x_i) \end{pmatrix}$$

log-sum-exp

- Ideas:
  1. use log-sum-exp to approximate max
  2. instead of counting number of times this quantity is positive, use this quantity as objective function

Q: What happens when W=0?

Multi-Class Logistic Regression

$$f(W) = \sum_{i=1}^{n} -w_{y_i}^T x_i + \log\left(\sum_{c=1}^{k} \exp(w_c^T x_i)\right)$$

"Softmax loss"

$$f(W) = \sum_{i=1}^{n} -w_{y_i}^T x_i + \log\left(\sum_{c=1}^{k} \exp(w_c^T x_i)\right) + \frac{\lambda}{2}\sum_{c=1}^{k}\sum_{j=1}^{d} w_{cj}^2$$

L2-regularized softmax loss

- Multi-class (multinomial) logistic regression: optimize W with L2-regularized softmax loss

Multi-Class Logistic Regression

$$f(W) = \sum_{i=1}^{n}\Big[\underbrace{-w_{y_i}^T x_i}_{\text{Tries to make } w_c^T x_i \text{ big for the correct label}} + \underbrace{\log\Big(\sum_{c=1}^{k} \exp(w_c^T x_i)\Big)}_{\text{Approximates } \max_c \{w_c^T x_i\} \text{ so tries to make } w_c^T x_i \text{ small for all labels.}}\Big] + \underbrace{\frac{\lambda}{2}\sum_{c=1}^{k}\sum_{j=1}^{d} w_{cj}^2}_{\text{Usual } L_2\text{-regularizer on elements of 'W'}}$$

- This objective is convex (should be clear for 1st and 3rd terms).
  - It's differentiable so you can use gradient descent.
- When k=2, equivalent to using binary logistic loss.
  - Not obvious at the moment.

Softmax Function

$$\text{Softmax} : \mathbb{R} \times \mathbb{R}^K \to (0, 1)$$

$$\text{Softmax}\Big(\underbrace{z, \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_K \end{bmatrix}}_{z \in \begin{bmatrix} z_1 \\ \vdots \\ z_K \end{bmatrix}}\Big) = \frac{\exp(z)}{\sum_{c=1}^{K} \exp(z_c)}$$

- Evaluates "max-ness" of z compared to group
  - if z is big compared to every other z_c, softmax is close to 1

"Max-ness" to Find "Cat-ness"

$$P(y_i = \text{"cat"} \mid W, \cdot) : \mathbb{R}^d \rightarrow (0, 1)$$

$$P(y_i = \text{"cat"} \mid W, x_i) = \text{Softmax}\left(w_{cat}^T x_i, \begin{bmatrix} w_{cat}^T x_i \\ w_{dog}^T x_i \\ w_{human}^T x_i \end{bmatrix}\right)$$

"cat-ness"

"How big is my cat score compared to my dog score and human score?"

$w_{cat}^T x_i = 1.83$

$w_{dog}^T x_i = -1.17$

$w_{human}^T x_i = -2.20$

cat-ness = ~93.6%

dog-ness = ~0.04%

human-ness = ~0.02%

Multi-Class Linear Prediction in Matrix Notation

- In multi-class linear classifiers our weights are:

$$W = \underbrace{\begin{bmatrix} \text{——} & w_1^T & \text{——} \\ \text{——} & w_2^T & \text{——} \\ & \vdots & \\ \text{——} & w_k^T & \text{——} \end{bmatrix}}_{d} \Big\} k$$

- To predict on all training examples, we first compute all $w_c^T x_i$.
  - Or in matrix notation:

$$\underbrace{\begin{bmatrix} w_1^T x_1 & w_2^T x_1 & \cdots & w_k^T x_1 \\ w_1^T x_2 & w_2^T x_2 & \cdots & w_k^T x_2 \\ & \vdots & & \\ w_1^T x_n & w_2^T x_2 & \cdots & w_k^T x_n \end{bmatrix}}_{XW^T} = \underbrace{\begin{bmatrix} \text{——} & x_1^T & \text{——} \\ \text{——} & x_2^T & \text{——} \\ & \vdots & \\ \text{——} & x_n^T & \text{——} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_k \\ | & | & & | \end{bmatrix}}_{W^T}$$

  - So predictions are maximum column indices of XW^T (which is 'n' by 'k').

How Do I Regularize W?

- The Frobenius norm of a (‘k’ by ‘d’) matrix ‘W’ is defined by:

$$\|W\|_F = \sqrt{\sum_{c=1}^{k}\sum_{j=1}^{d} w_{jc}^2}$$

(L2-norm if you "stack" elements into one big vector)

- We can use this to write regularizer in matrix notation:

$$\frac{\lambda}{2}\sum_{c=1}^{k}\sum_{j=1}^{d} w_{cj}^2 = \frac{\lambda}{2}\sum_{c=1}^{k}\|w_c\|^2$$ ("L2-regularizer on each vector")

$$= \frac{\lambda}{2}\|W\|_F^2$$ ("Frobenius-regularizer on matrix")

Summary

- Sigmoid function: turn linear predictions into probabilities.
- One vs all: turn binary classifiers into a multi-class classifier.
- Multi-class SVM: measure violation of classification constraints.
- Multi-class logistic regression: log-sum-exp approximation of “correct=max” constraint violations
- Next time: feature engineering and how to represent text data

Review Questions

- Q1: How does the sigmoid function satisfy the definition of probability?
- Q2: What makes the one-vs-all classifier under-constrained?
- Q3: Mathematically, what leads to the “sum”-rule and “max”-rule variants of multi-class SVM?
- Q4: How do we know that the objective functions for multi-class SVM and multi-class logistic regression are convex?

“All-Pairs” and ECOC Classification

- Alternative to “one vs. all” to convert binary classifier to multi-class is “all pairs”.
  - For each pair of labels ‘c’ and ‘d’, fit a classifier that predicts +1 for examples of class ‘c’ and -1 for examples of class ‘d’ (so each classifier only trains on examples from two classes).
  - To make prediction, take a vote of how many of the (k-1) classifiers for class ‘c’ predict +1.
  - Often works better than “one vs. all”, but not so fun for large ‘k’.

- A variation on this is using “error correcting output codes” from information theory (see Math 342).
  - Each classifier trains to predict +1 for some of the classes and -1 for others.
  - You setup the +1/-1 code so that it has an “error correcting” property.
    - It will make the right decision even if some of the classifiers are wrong.

Motivation: Dog Image Classification

- Suppose we're classifying images of dogs into breeds:

- What if we have images where class label isn't obvious?
  - Siberian husky vs. Inuit dog?

https://www.slideshare.net/angjoo/dog-breed-classification-using-part-localization
https://ischlag.github.io/2016/04/05/important-ILSVRC-achievements

Learning with Preferences

- Do we need to throw out images where label is ambiguous?
  - We don’t have the y_i.

  - We want classifier to prefer Siberian husky over bulldog, Chihuahua, etc.
    - Even though we don’t know if these are Siberian huskies or Inuit dogs.
  - Can we design a loss that enforces preferences rather than “true” labels?

https://ischlag.github.io/2016/04/05/important-ILSVRC-achievements

Learning with Pairwise Preferences (Ranking)

- Instead of y_i, we're given list of (c_1,c_2) preferences for each 'i':

We want $w_{c_1}^T x_i > w_{c_2}^T x_i$ for these particular (c_1, c_2) values

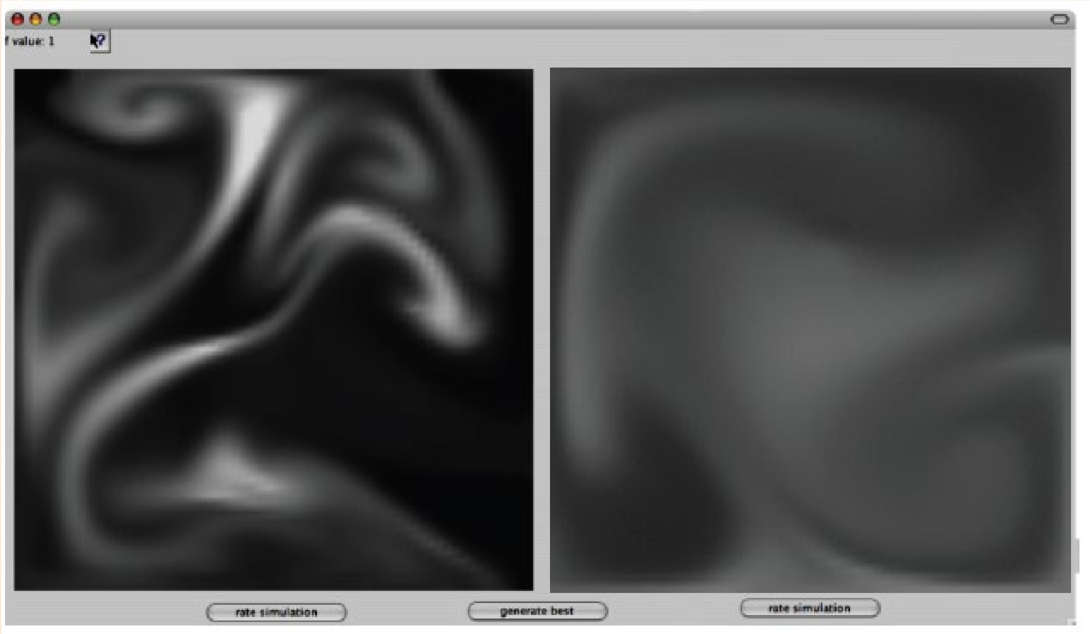
- Multi-class classification is special case of choosing (y_i,c) for all 'c'.

- By following the earlier steps, we can get objectives for this setting:

$$\sum_{i=1}^{n} \underbrace{\sum_{(c_1,c_2)} \max\{0, 1 - w_{c_1}^T x_i + w_{c_2}^T x_i\}}_{\text{"sum" version of multi-class SVM}} + \frac{\lambda}{2}\|W\|_F^2$$

Learning with Pairwise Preferences (Ranking)

- Pairwise preferences for computer graphics:
  - We have a smoke simulator, with several parameters:

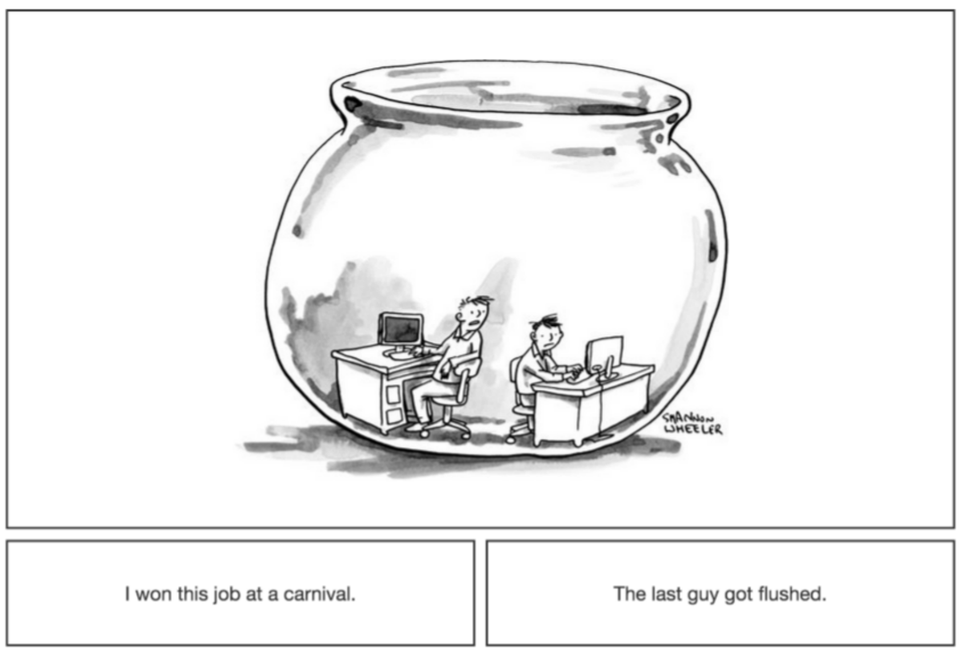


  - Don't know what the optimal parameters are, but we can ask the artist:
    - "Which one looks more like smoke"?

https://circle.ubc.ca/bitstream/handle/2429/30519/ubc_2011_spring_brochu_eric.pdf?sequence=3

Learning with Pairwise Preferences (Ranking)

- Pairwise preferences for humour:
  - New Yorker caption contest:



  - “Which one is funnier”?

https://homes.cs.washington.edu/~jamieson/resources/next.pdf

Risk Scores

- In medicine/law/finance, risk scores are sometimes used to give probabilities:

1.	**C**ongestive Heart Failure	1 point	...
2.	**H**ypertension	1 point	+ ...
3.	**A**ge $\geq$ 75	1 point	+ ...
4.	**D**iabetes Mellitus	1 point	+ ...
5.	Prior **S**troke or Transient Ischemic Attack	**2** points	+
		SCORE	=

SCORE	0	1	2	3	4	5	6
RISK	1.9%	2.8%	4.0%	5.9%	8.5%	12.5%	18.2%

Figure 1: CHADS$_2$ risk score of Gage et al. (2001) to assess stroke risk (see www.mdcalc.com for other medical scoring systems). The variables and points of this model were determined by a panel of experts, and the risk estimates were computed empirically from data.

  - Get integer-valued "points" for each "risk factor", and probability is computed from data based on people with same number of points.
  - Less accurate than fancy models, but interpretable and can be done by hand.
    - Some work on trying to "learn" the whole thing (like doing feature selection then rounding).

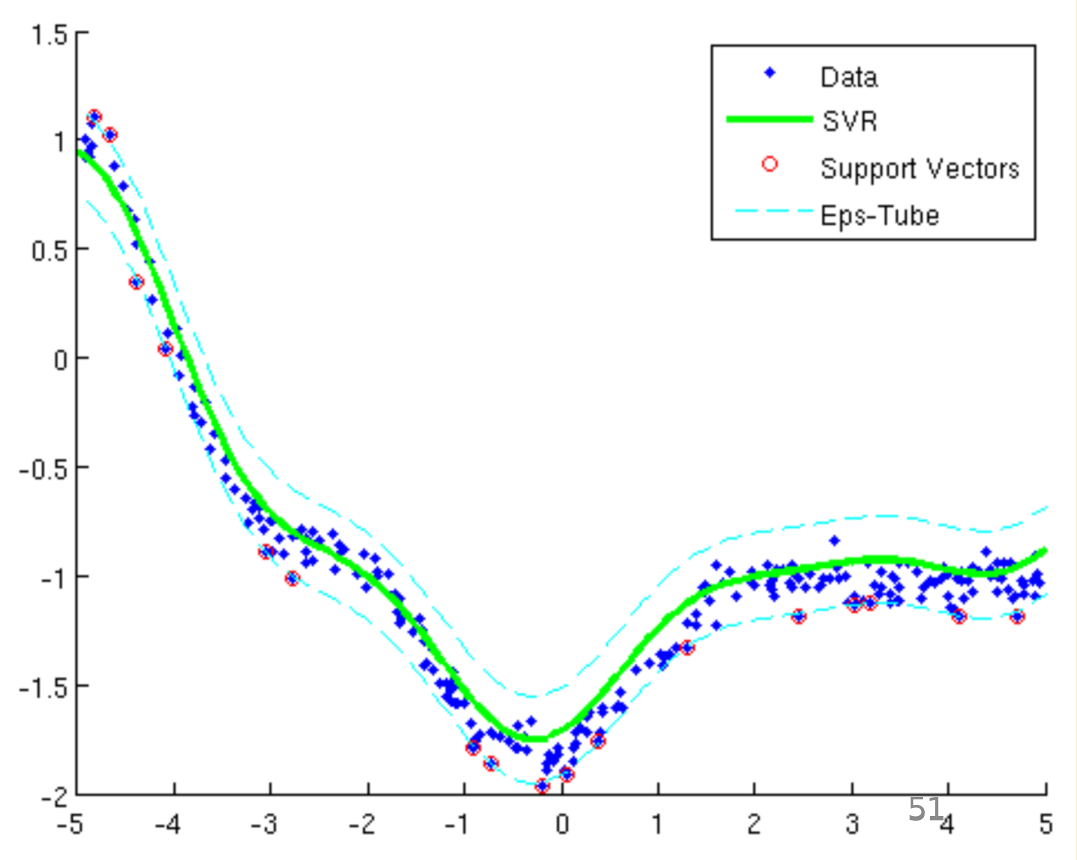
https://arxiv.org/pdf/1610.00168.pdf

Support Vector Regression

- Support vector regression objective (with hyper-parameter ϵ):

$$f(w) = \sum_{i=1}^{n} \max\{0, |w^T x_i - y_i| - \epsilon\} + \frac{\lambda}{2}\|w\|^2$$

  - Looks like L2-regularized robust regression with the L1-loss.
  - But have loss of 0 if $\hat{y}_i$ within ϵ of $\tilde{y}_i$.
    - So doesn't try to fit data exactly.
      - This can help fight overfitting.
  - Support vectors are points with loss>0.
    - Points outside the "epsilon-tube".
  - Example with Gaussian-RBFs as features:

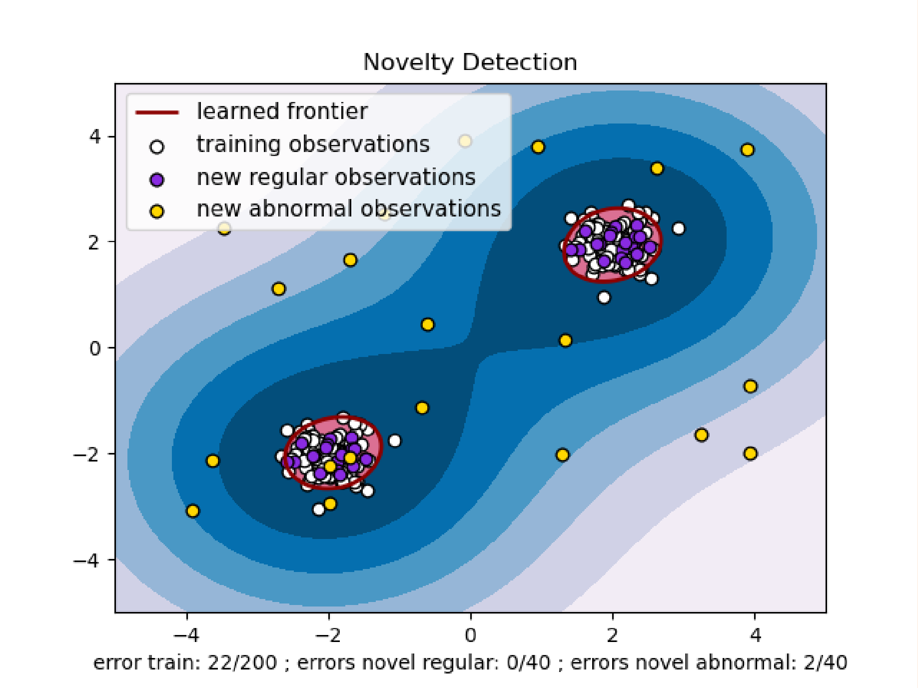


1-Class SVMs

- 1-class SVMs for outlier detection.

$$f(w, w_0) = \sum_{i=1}^{N} [\max\{0, w_0 - w^T x_i\} - w_0] + \frac{\lambda}{2}\|w\|_2^2$$

  - Variables are ‘w’ (vector) and ‘w_0’ (scalar).
  - Only trains on “inliers”.
    - Tries to make $w^T x_i$ bigger than w_0 for inliers.
    - At test time: says “outlier” if $w^T x_i < w_0$.
    - Usually used with RBFs.



https://scikit-learn.org/stable/auto_examples/svm/plot_oneclass.html