

Entropy := measure of heterogeneity of values

$$= - \sum_c P_c \log P_c$$

↑
frequency of
value k .



CPSC 340: Machine Learning and Data Mining

Fundamentals of Learning
Summer 2021

REMINDER TO HIT RECORD

In This Lecture

- **Applying Decision Trees (10 minutes)**
- **Golden Rule of Machine Learning (10 minutes)**
- **Fundamental Trade-off (10 minutes)**
- **Hyper-Parameters (10 minutes)**

Coming Up Next

APPLYING DECISION TREES

Supervised Learning Application

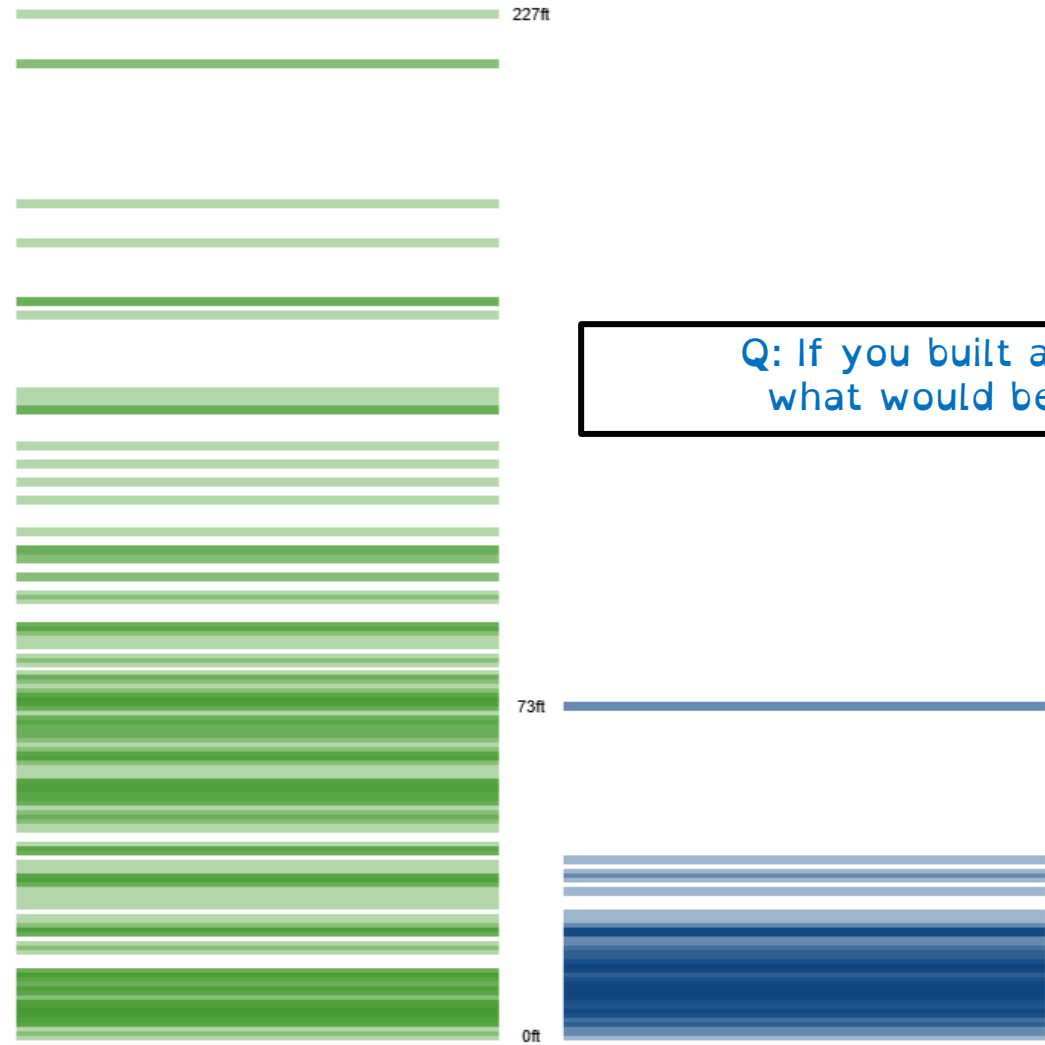
- We motivated supervised learning by the “food allergy” example.
- But we can use supervised learning for any input -> output mapping.
 - Email spam filtering.
 - Optical character recognition on scanners.
 - Recognizing faces in pictures.
 - Recognizing tumours in medical images.
 - Speech recognition on phones.
 - Your problem in industry/research?

Motivation: Determine Home City

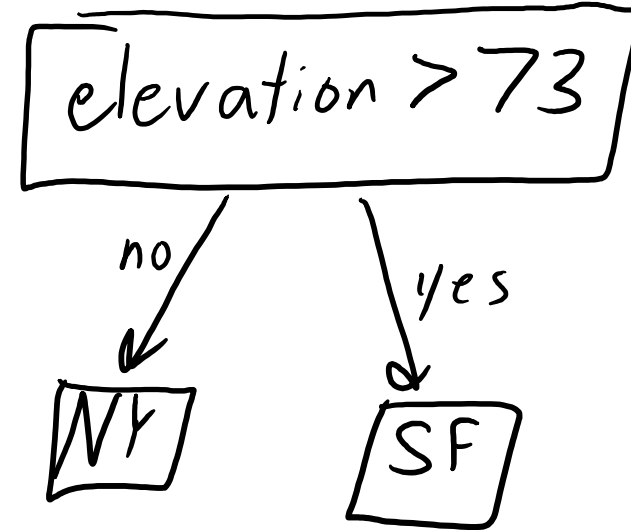
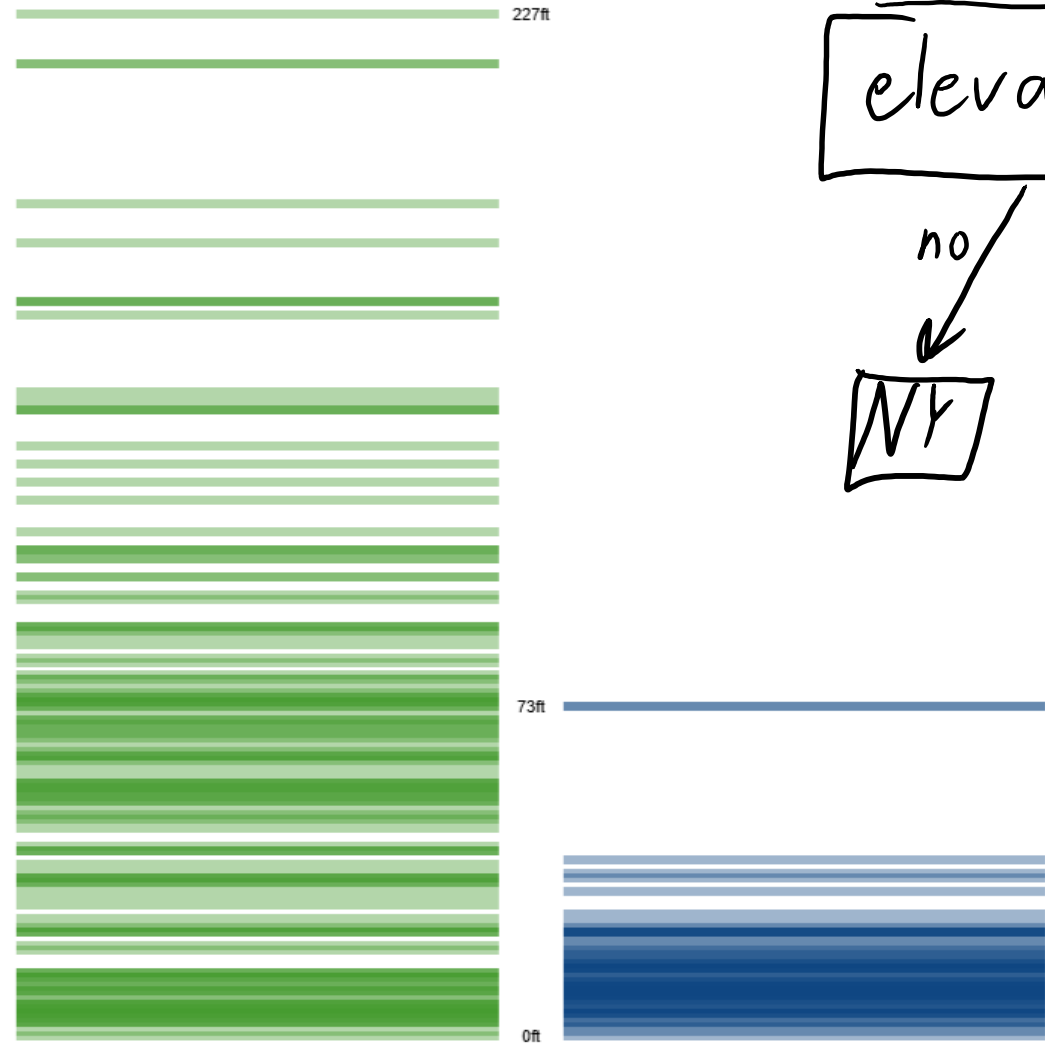
- We are given data from 248 homes.
- For each home/example, we have these features:
 - Elevation.
 - Year.
 - Bathrooms
 - Bedrooms.
 - Price.
 - Square feet.
- Goal is to build a program that predicts **SF** or **NY**.

Step 1: Look at the Data!

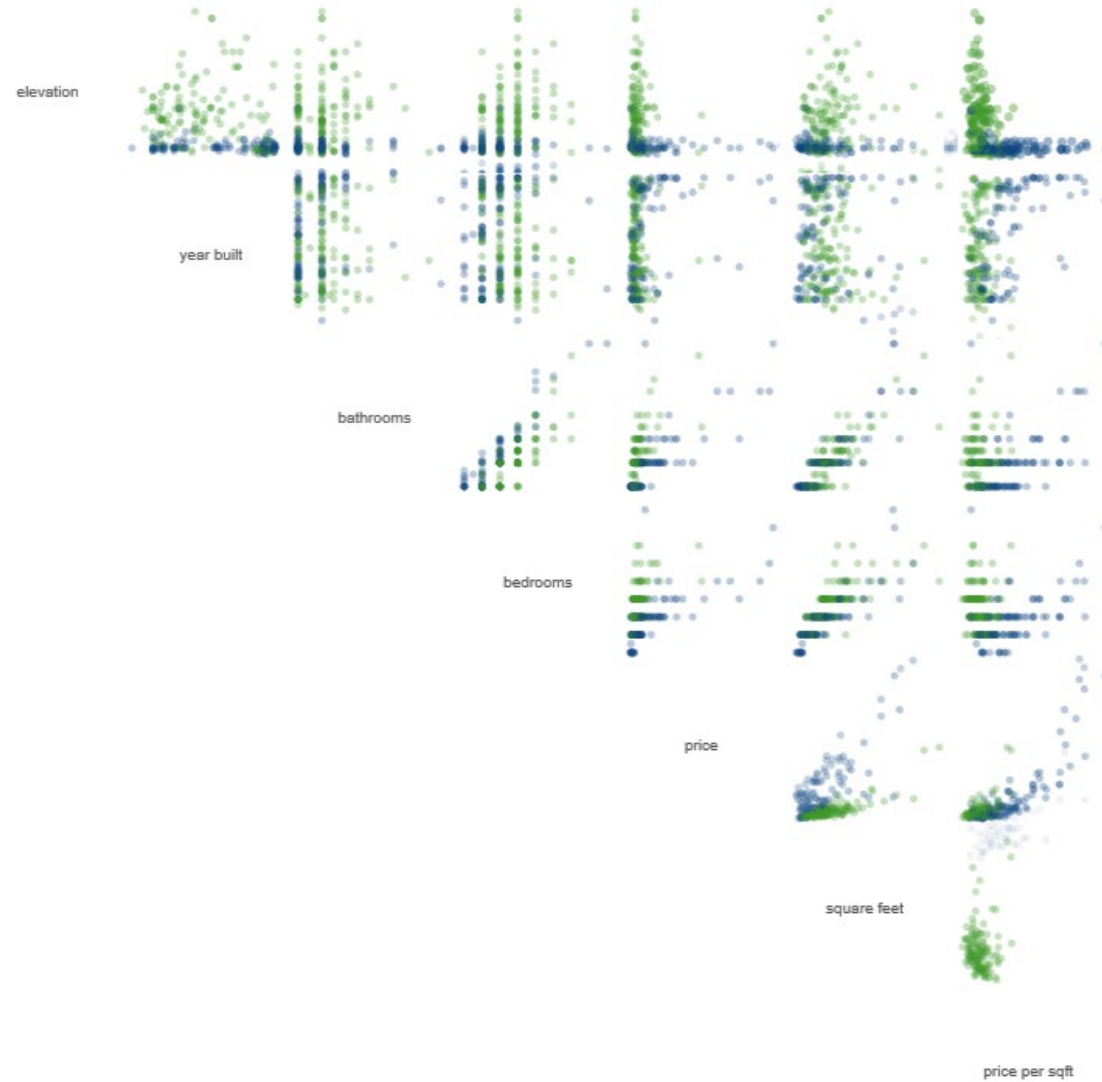
Plotting Elevation



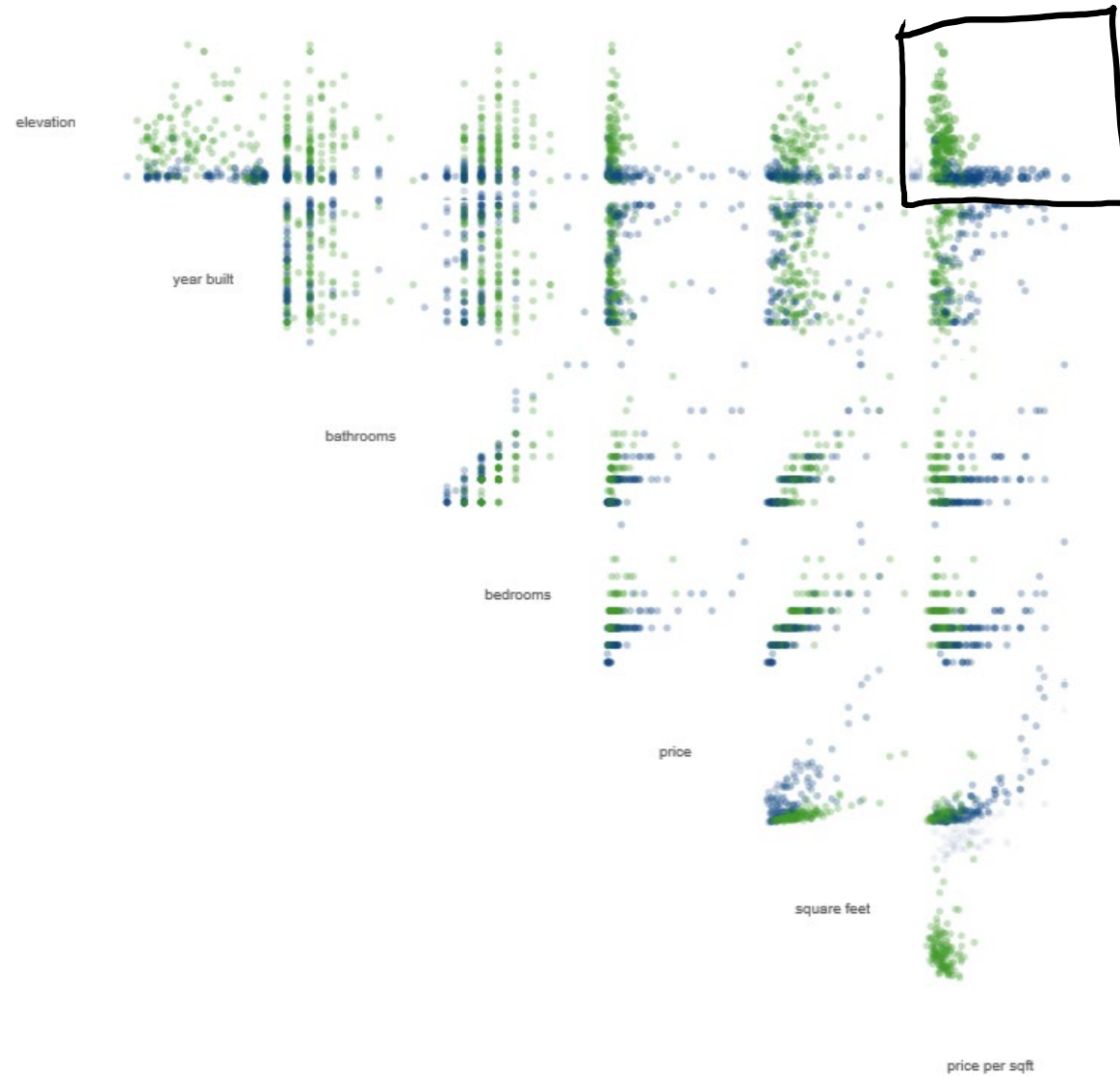
Simple Decision Stump



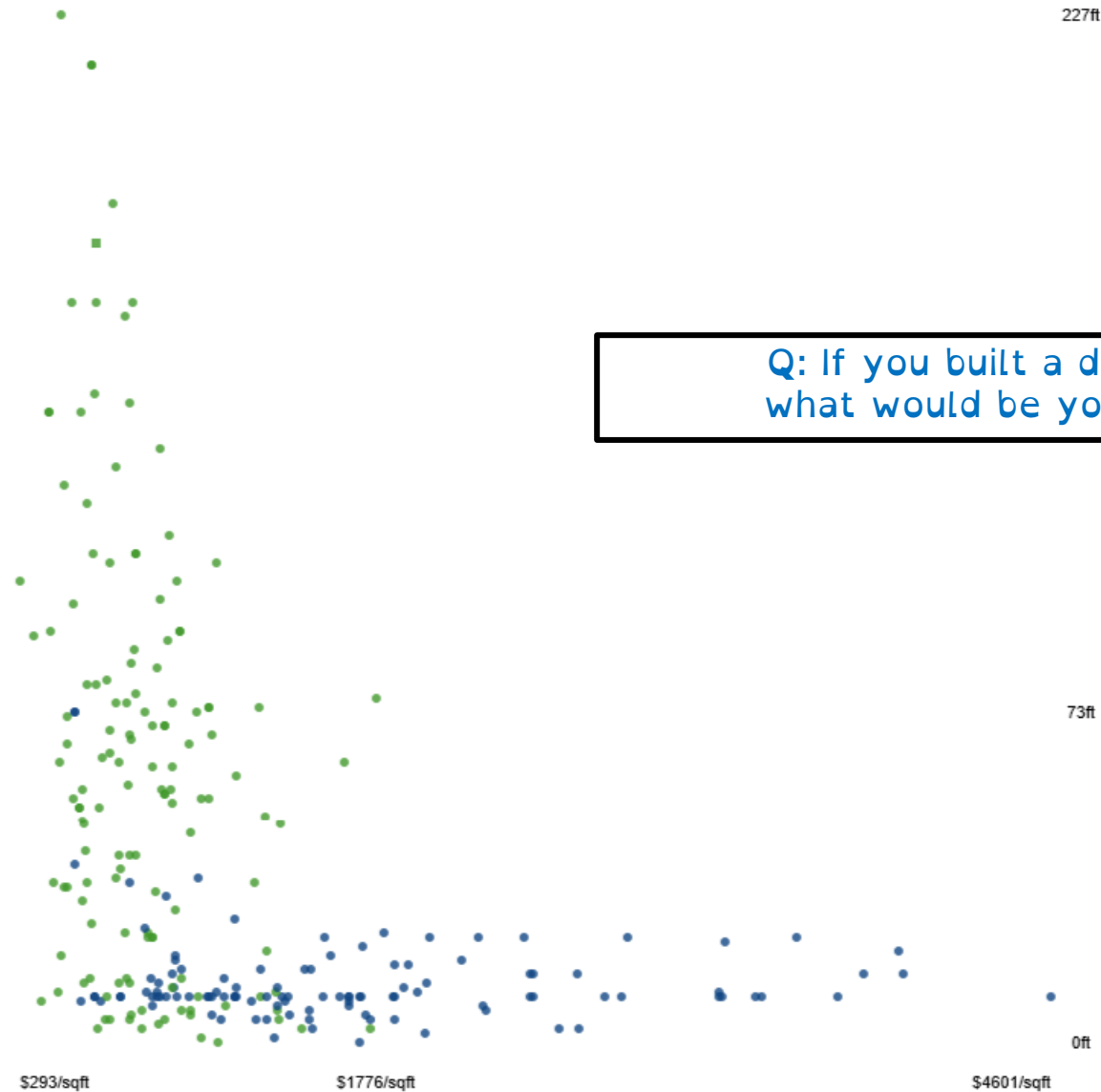
Scatterplot Array



Scatterplot Array

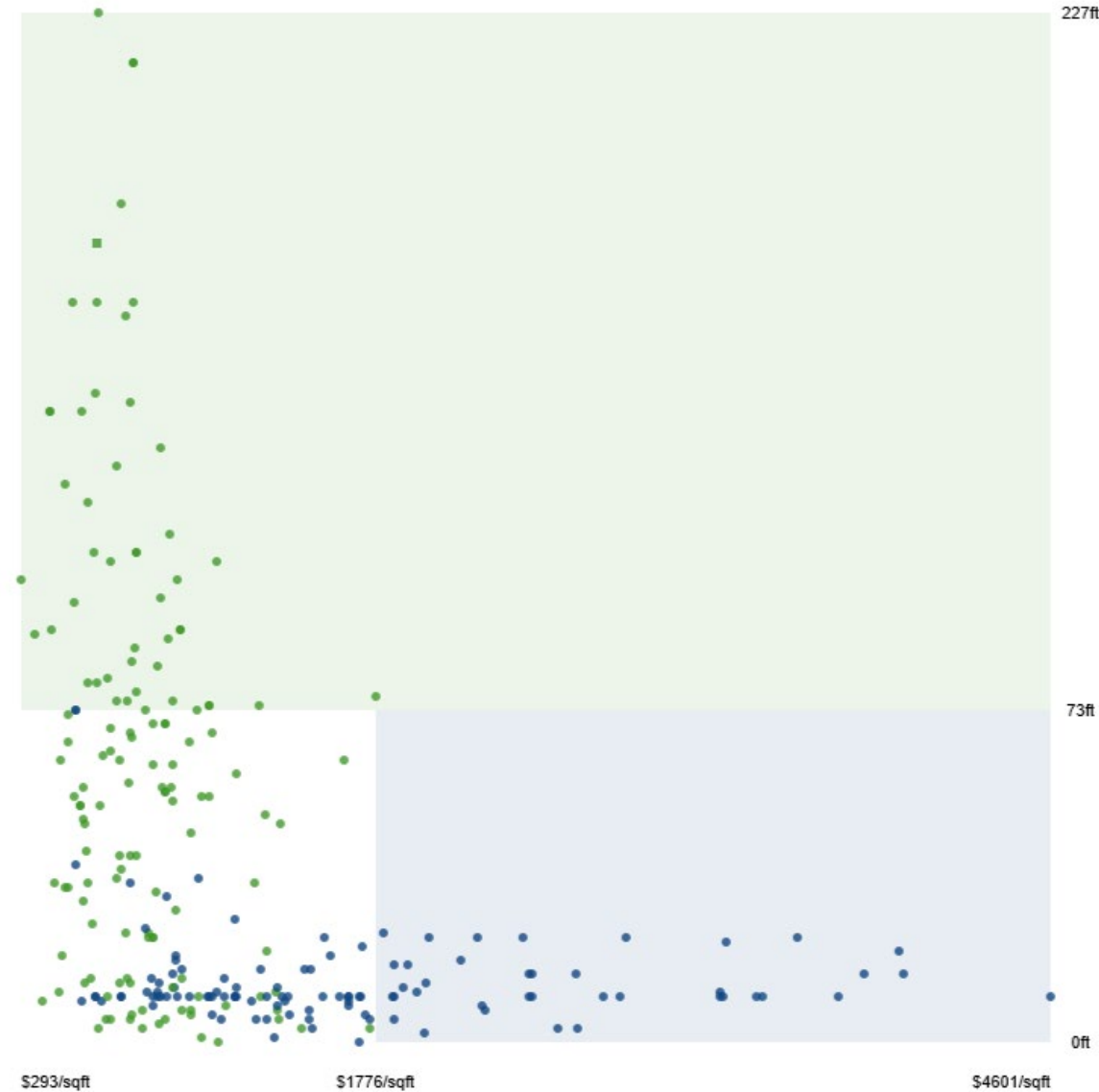


Plotting Elevation and Price/SqFt

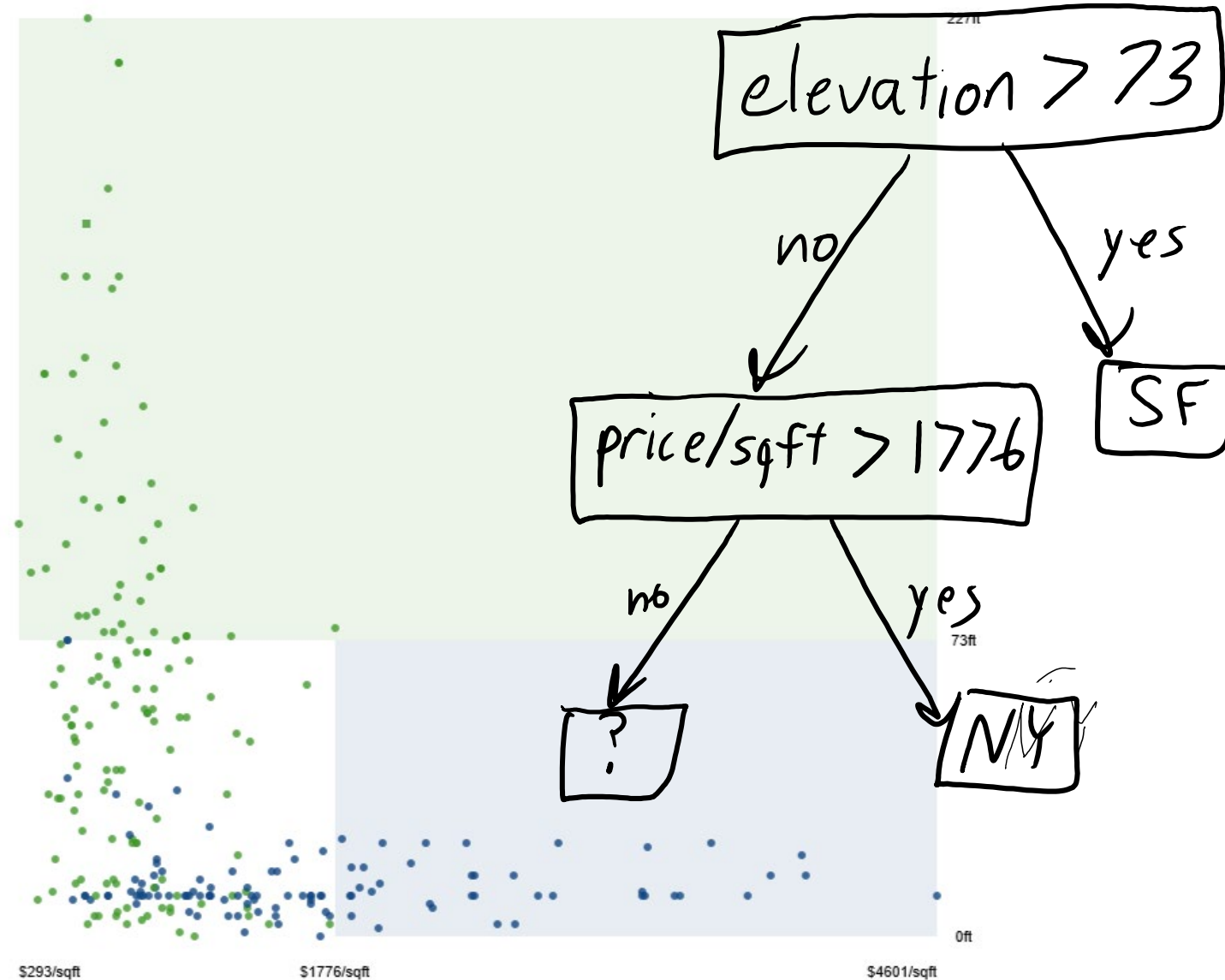


Q: If you built a decision tree here, what would be your splitting rules?

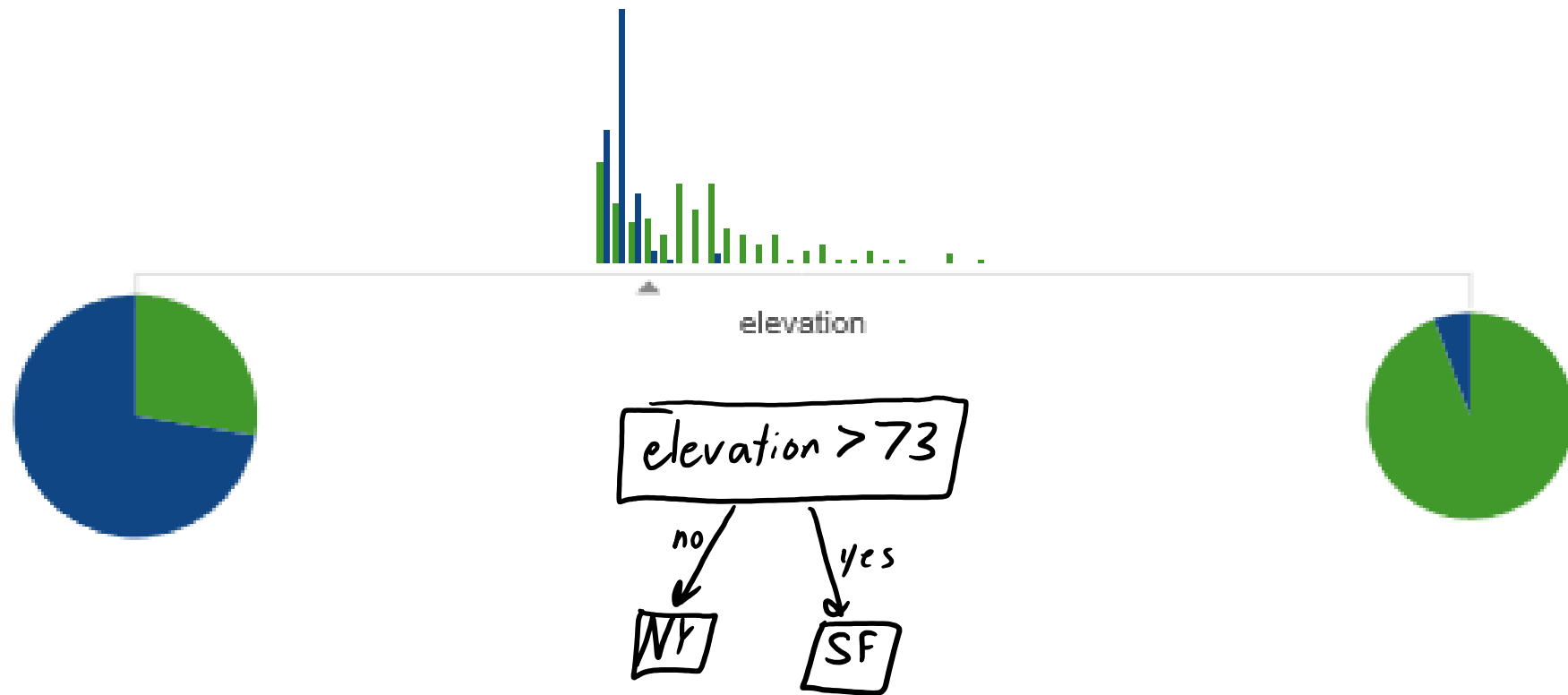
Simple Decision Tree Classification



Simple Decision Tree Classification

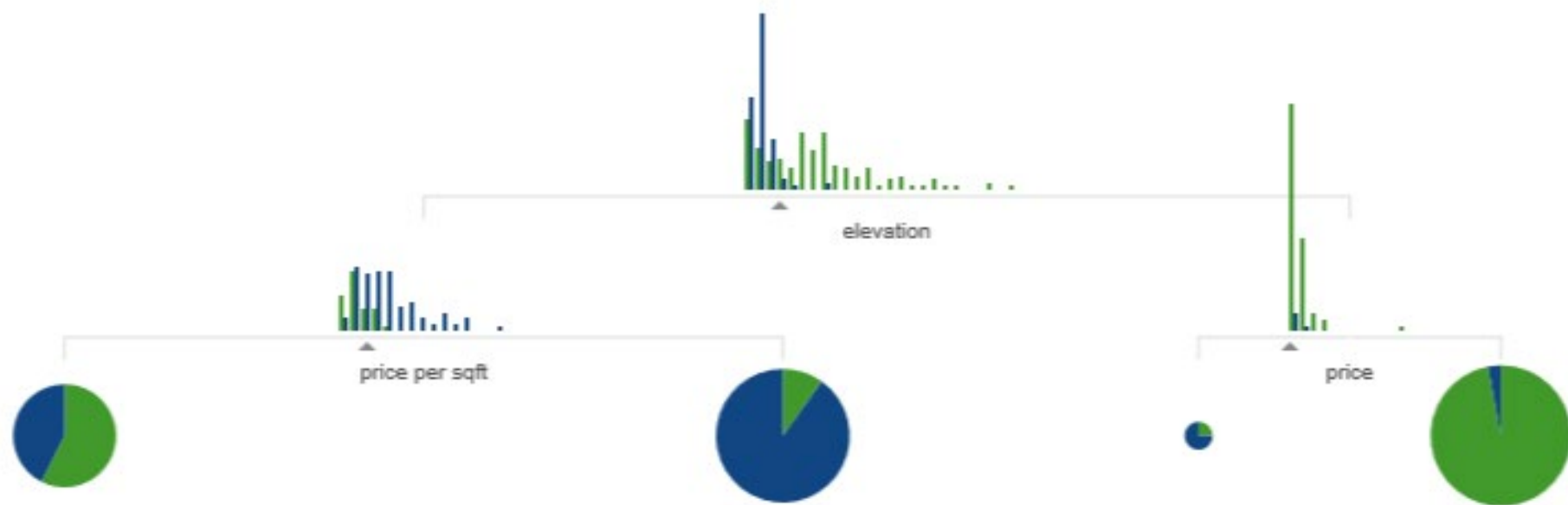


How does the depth affect accuracy?



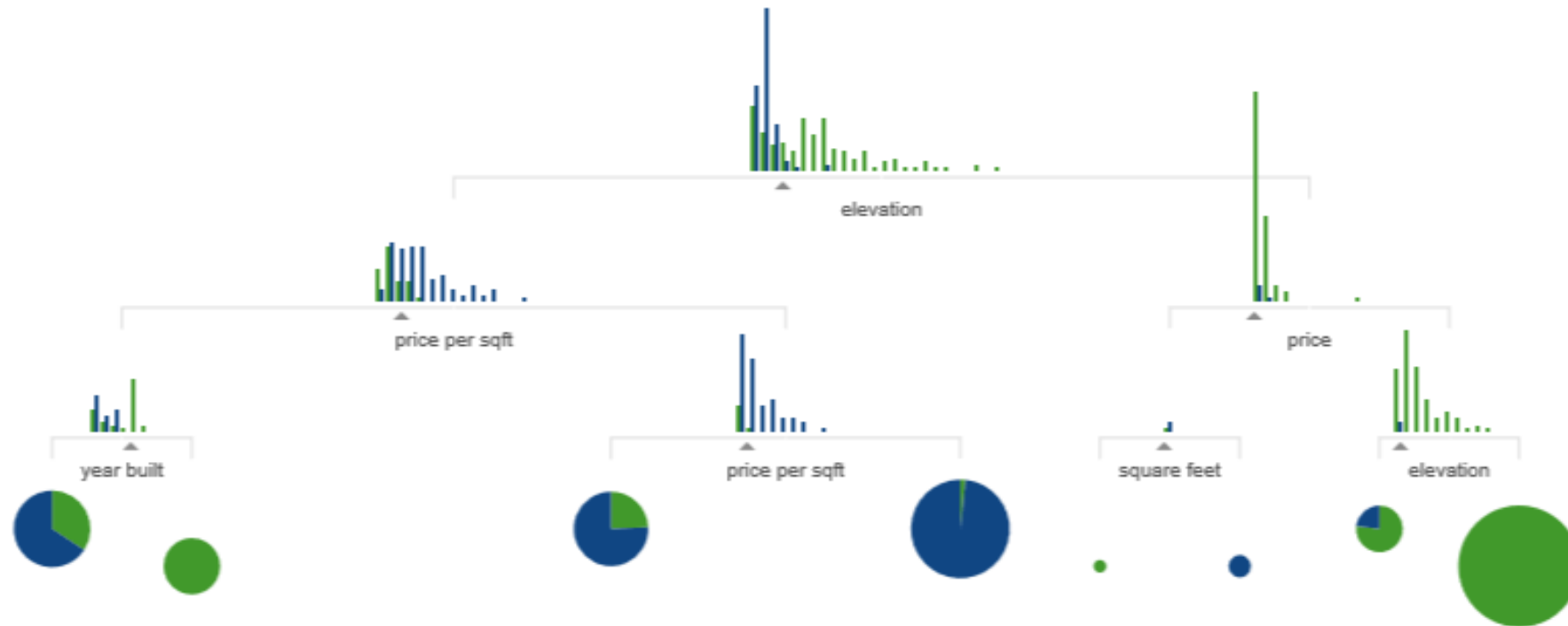
This is a good start (> 75% accuracy).

How does the depth affect accuracy?



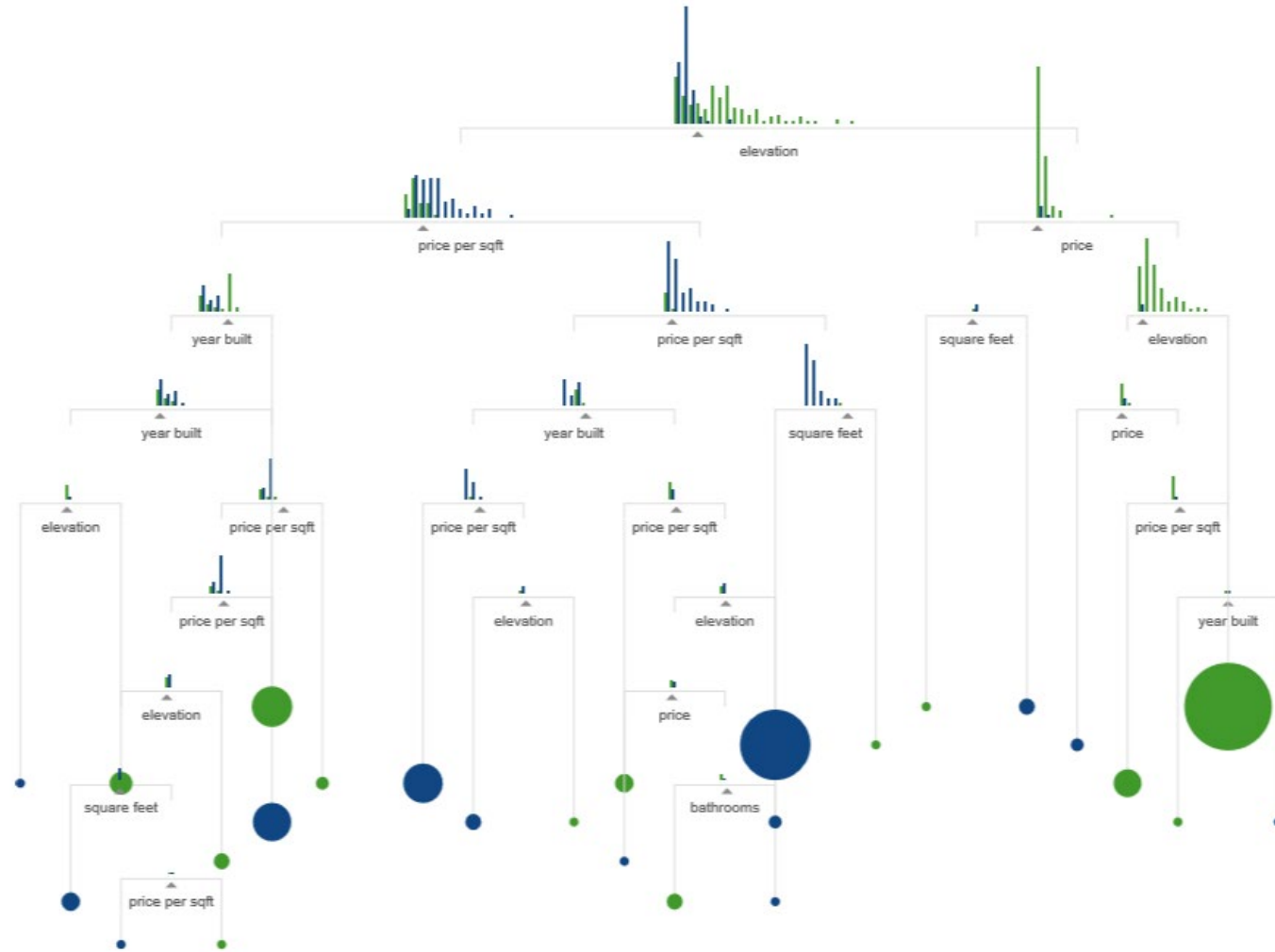
Start splitting the data recursively...

How does the depth affect accuracy?



Accuracy keeps increasing as we add depth.

How does the depth affect accuracy?



Eventually, we can perfectly classify all of our data.

Training vs. Testing Error

- With this decision tree, ‘**training accuracy**’ is 1.
 - It **perfectly labels the data we used** to make the tree.
- We are now given features for **217 new homes**.
- What is the ‘**testing accuracy**’ on the new data?
 - How does it do **on data not used** to make the tree?



- **Overfitting:**
 - Our **rules got too specific to our exact training dataset**.
 - Some of the “deep” splits only use a few examples (bad “coupon collecting”).

Supervised Learning Notation

- Recall: we are given **training data** where we know labels:

$X =$	Egg	Milk	Fish	Wheat	Shellfish	Peanuts	...	$y =$	Sick?
	0	0.7	0	0.3	0	0			1
	0.3	0.7	0	0.6	0	0.01			1
	0	0	0	0.8	0	0			0
	0.3	0.7	1.2	0	0.10	0.01			1
0.3	0	1.2	0.3	0.10	0.01		1		

- But there is also **testing data** we want to label:

$\tilde{X} =$ <i>t x d</i>	Egg	Milk	Fish	Wheat	Shellfish	Peanuts	...	$\tilde{y} =$ <i>t x 1</i>	Sick?
	0.5	0	1	0.6	2	1			?
	0	0.7	0	1	0	0			?
3	1	0	0.5	0	0		?		

Supervised Learning Notation

- Recall: typical supervised learning steps:
 1. Build **model based on training data** X and y (training phase).
 2. Model makes **predictions \hat{y} on test data \tilde{X}** (testing phase).
- Instead of **training error**, consider **test error**:
 - Are predictions \hat{y} similar to true unseen labels \tilde{y} ?

When you test your model



Coming Up Next

GOLDEN RULE OF MACHINE LEARNING

Recall: Learning

“...learning refers to the process of extracting structure—**statistical regularities**—from input data, and encoding that structure into the parameters...” [Zador 2019]

- An unspoken rule of ML:
If your solution **doesn't generalize**, it didn't learn well

Q: What does it mean for my learned solution to generalize?

- Good Learning:
train in order to **perform well on test!**

Goal of Machine Learning

- In machine learning:
 - What we care about is the test error!
- Midterm analogy:
 - Training error: how you do on practice midterm.
 - Test error: how you do on actual midterm.
 - Goal: do well on actual midterm, not the practice one.
- Memorization vs Learning:
 - Can do well on training data by memorizing it.
 - Just build a dictionary to get 100% accuracy!
 - You've only learned if you can do well in new situations.

Golden Rule of Machine Learning

- Even though what we care about is test error:
 - THE TEST DATA CANNOT INFLUENCE THE TRAINING PHASE IN ANY WAY.
- We're measuring test error to see how well we do on new data:
 - If used during training, doesn't measure this.
 - You can start to overfit if you use it during training.
 - Midterm analogy:
 - you saw the exam before it was released and memorized all the answers.

Golden Rule of Machine Learning

- Even though what we care about is test error:
 - THE TEST DATA CANNOT INFLUENCE THE TRAINING PHASE IN ANY WAY.



Tom Simonite

June 4, 2015

Why and How Baidu Cheated an Artificial Intelligence Test

Machine learning gets its first cheating scandal.

The sport of training software to act intelligently just got its first cheating scandal. Last month Chinese search company Baidu announced that its image recognition software had [inched ahead of Google's on a standardized](#)

Golden Rule of Machine Learning

- Even though what we care about is test error:
 - **THE TEST DATA CANNOT INFLUENCE THE TRAINING PHASE IN ANY WAY.**
- You also **shouldn't change the test set** to get the result you want.

DECEPTION AT DUKE: FRAUD IN CANCER CARE?

Were some cancer patients at Duke University given experimental treatments based on fabricated data? Scott Pelley reports.

- http://blogs.sciencemag.org/pipeline/archives/2015/01/14/the_dukepotti_scandal_from_the_inside

Digression: Golden Rule and Hypothesis Testing

- Note the **golden rule applies to hypothesis testing in scientific studies.**
 - **Data that you collect can't influence the hypotheses that you test.**
- **EXTREMELY COMMON** and a **MAJOR PROBLEM**, coming in many forms:
 - Collect more data until you coincidentally get significance level you want.
 - Try different ways to measure performance, choose the one that looks best.
 - Choose a different type of model/hypothesis after looking at the test data.
- **If you want to modify your hypotheses, you need to test on new data.**
 - Or at least **be aware and honest about this issue** when reporting results.

Digression: Golden Rule and Hypothesis Testing

- Note the **golden rule applies to hypothesis testing in scientific studies.**
 - **Data that you collect can't influence the hypotheses that you test.**
- **EXTREMELY COMMON** and a **MAJOR PROBLEM**, coming in many forms:
 - “Replication crisis in Science”.
 - “Why Most Published Research Findings are False”.
 - “False-Positive Psychology: Undisclosed Flexibility in Data Collection and Analysis Allows Presenting Anything as Significant”.
 - “HARKing: Hypothesizing After the Results are Known”.
 - “Hack Your Way To Scientific Glory”.
 - “Psychology's Replication Crisis Has Made The Field Better” (some solutions)

When your complex model doesn't generalize



Coming Up Next

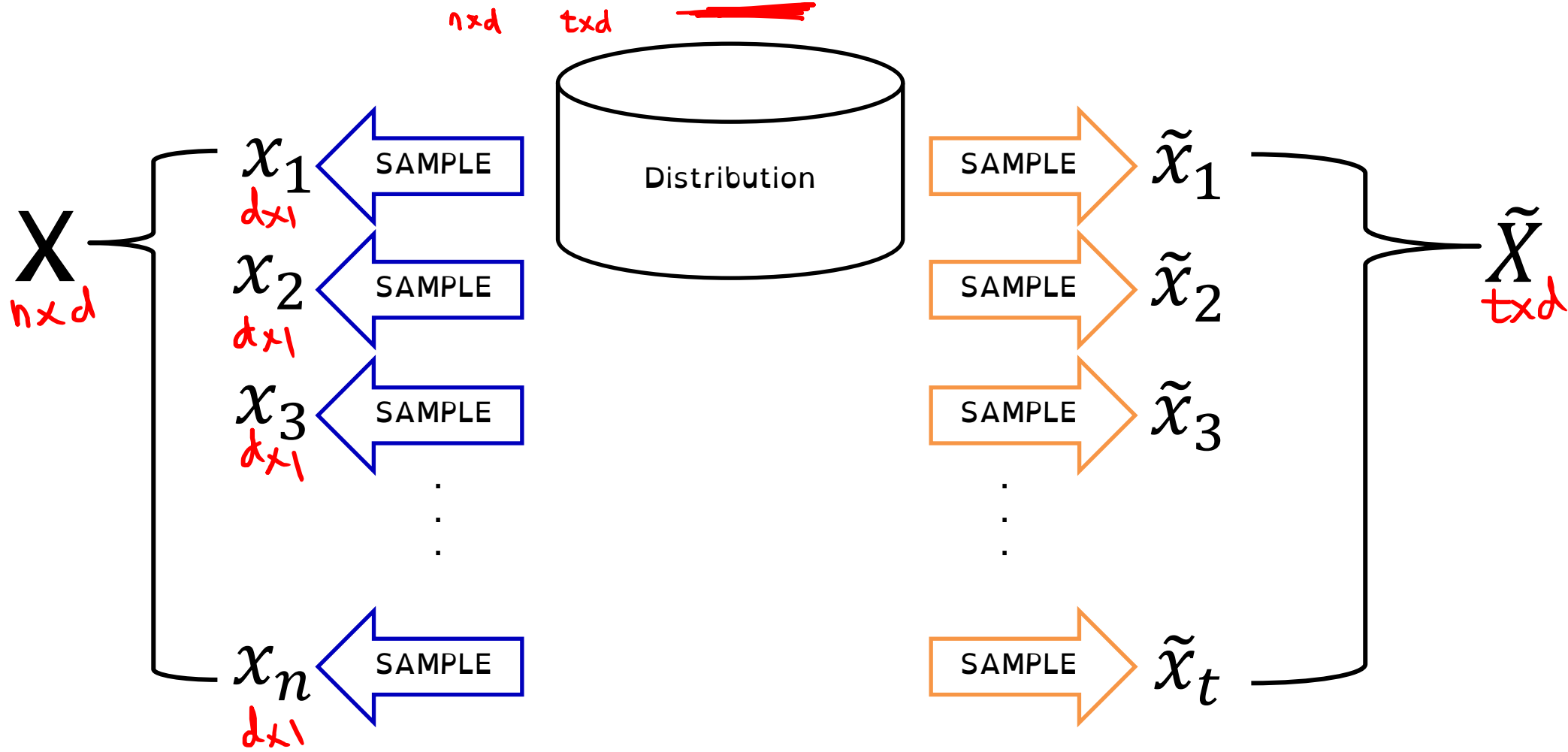
FUNDAMENTAL TRADE-OFF

Is Learning Possible?

- Does training error say anything about test error?
 - In general, NO: Test data might have nothing to do with training data.
 - E.g., study for CPSC 340 midterm, and we give you a history exam
- In order to learn, we need assumptions:
 - The training and test data need to be related in some way.
 - Most common assumption: independent and identically distributed (IID).

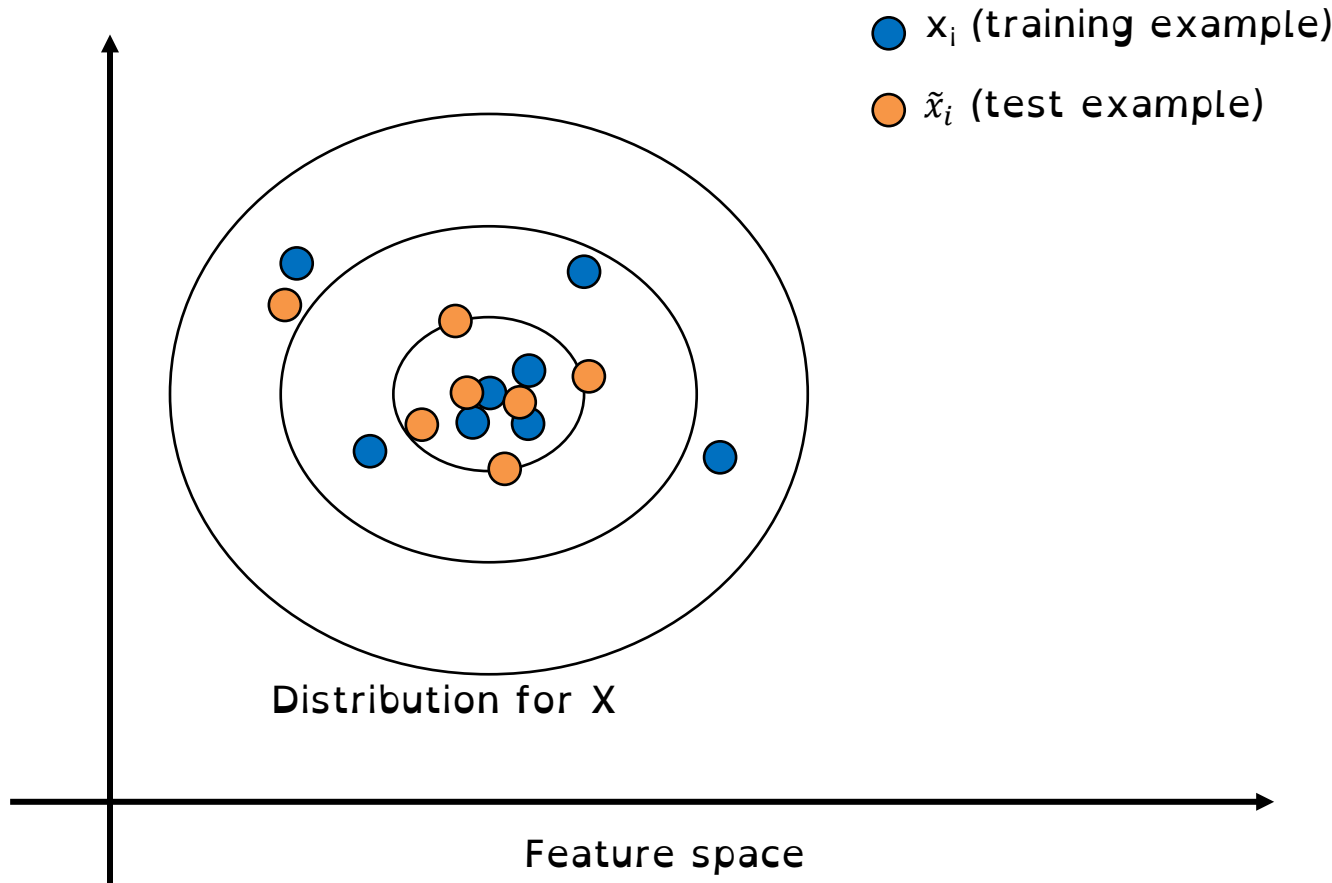
Distribution: A Data Generator

We will view X and \tilde{X} as samples collected from a distribution



Visualizing Distributions

Distribution := object that generates datapoints according to specified density.
Let's think of it as a "cloud" floating in the feature space.



IID Assumption

- Training/test data is independent and identically distributed (IID) if:
 - All examples come from the same distribution (identically distributed).
 - The examples are sampled independently (order doesn't matter).

Row 1 comes from same distribution as rows 2-3.

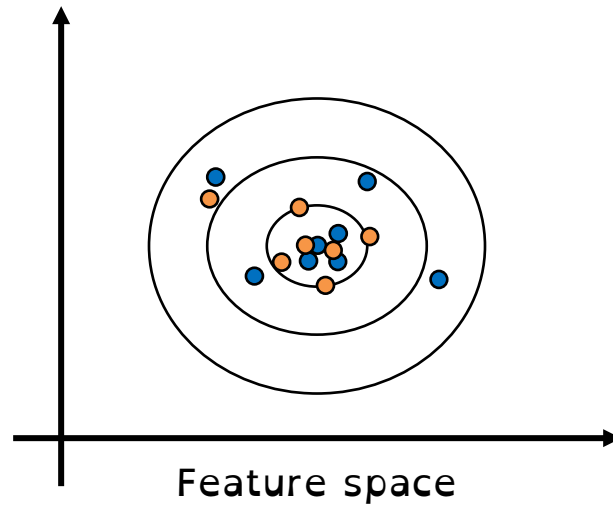
Age	Job?	City	Rating	Income
23	Yes	Van	A	22,000.00
23	Yes	Bur	BBB	21,000.00
22	No	Van	CC	0.00
25	Yes	Sur	AAA	57,000.00

Row 4 does not depend on values in rows 1-3.

- Examples in terms of cards:
 - Pick a card, put it back in the deck, re-shuffle, repeat. Is this IID? **Yes**
 - Pick a card, put it back in the deck, repeat. Is this IID? **No**
 - Pick a card, don't put it back, re-shuffle, repeat. Is this IID? **No**

IID Assumption

- Why does the IID assumption make learning possible?
 - Patterns in training examples are likely to be the same in test examples.



- The IID assumption is rarely true:
 - But it is often a good approximation.
 - There are other possible assumptions.
- Also, we're assuming IID across examples but not across features.

Learning Theory

- **Learning theory** explores how training error is related to test error.
- We'll look at a simple example, using this notation:
 - E_{train} is the **error on training data**.
 - E_{test} is the **error on testing data**.

Fundamental Trade-Off

- Start with $E_{\text{test}} = E_{\text{test}}$, then subtract and add E_{train} on the right:

$$E_{\text{test}} = (E_{\text{test}} - E_{\text{train}}) + E_{\text{train}}$$

"test error" "approximation error" "training error"
 E_{approx}

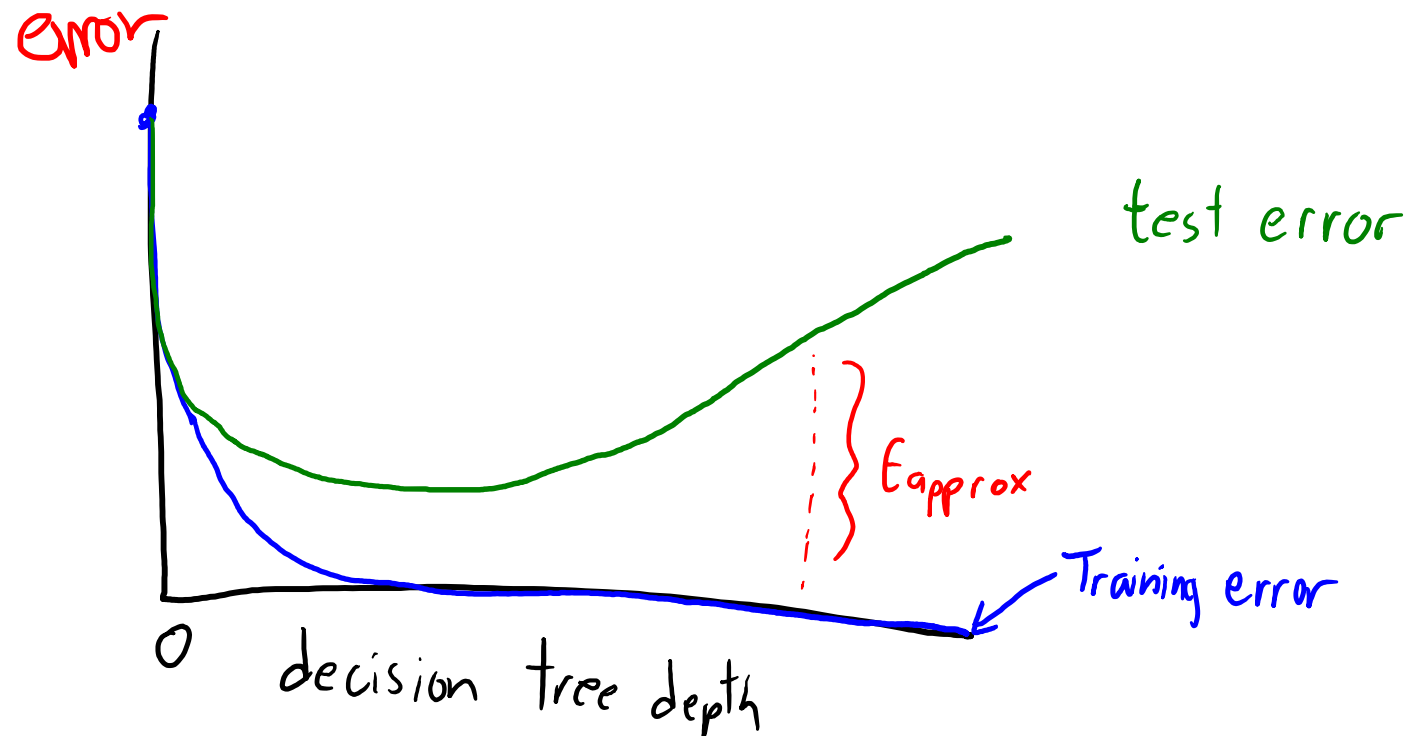
- How does this help?
 - If E_{approx} is (small/big), then E_{train} is a good approximation to E_{test} .
- What does E_{approx} ("amount of overfitting") depend on?
 - It tends to get smaller as n gets larger.
 - It tends to grow as model gets more complex.

Fundamental Trade-Off

- This leads to a **fundamental trade-off**:
 1. E_{train} : how small you can make the training error.
 - VS.
 2. E_{approx} : how well training error approximates the test error.
- **Simple models** (e.g. decision stumps):
 - E_{approx} is low (not very sensitive to training set).
 - But E_{train} might be high.
- **Complex models** (e.g. deep decision trees):
 - E_{train} can be low.
 - But E_{approx} might be high (very sensitive to training set).

Fundamental Trade-Off

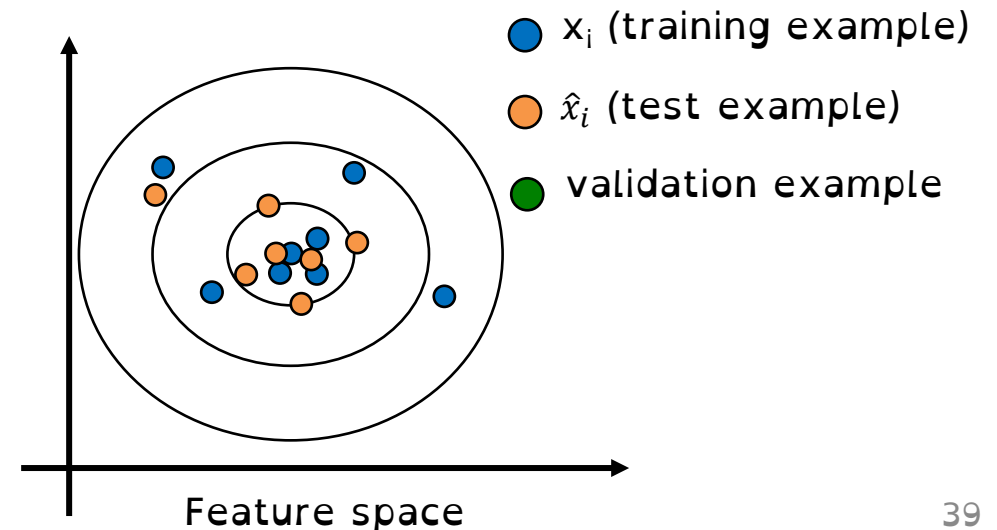
- Training error vs. test error for choosing depth:
 - Training error is high for low depth (**underfitting**)
 - Training error gets better with depth.
 - Test error initially goes down, but eventually increases (**overfitting**).



Validation Error

- How do we decide decision tree depth?
- We care about test error.
- But we can't look at test data.
- So what do we do?????

- One answer: Use part of the training data to approximate test error.
- Split training examples into training set and validation set:
 - Train model based on the training data.
 - Test model based on the validation data.



Validation Error

$$X = \begin{bmatrix} X_{\text{train}} \\ \text{---} \\ X_{\text{validate}} \end{bmatrix} \quad Y = \begin{bmatrix} Y_{\text{train}} \\ \text{---} \\ Y_{\text{validate}} \end{bmatrix}$$

} "train"
} "validation"

Step 1 is training: $\text{model} = \text{train}(X_{\text{train}}, Y_{\text{train}})$

Step 2 is predicting: $\hat{y} = \text{predict}(\text{model}, X_{\text{validate}})$

Step 3 is validating: $\text{error} = \text{sum}(\hat{y} \neq Y_{\text{validate}})$

Note: if examples are ordered, split should be random. validation error

Validation Error

- IID data: validation error is unbiased approximation of test error.

$$\underbrace{E}_{\substack{\text{Expectation} \\ \text{over IID} \\ \text{samples}}} [\underbrace{E_{\text{valid}}}_{\substack{\text{validation} \\ \text{error}}}] = \underbrace{E}_{\substack{\text{Expectation} \\ \text{over IID} \\ \text{samples}}} [\underbrace{E_{\text{test}}}_{\substack{\text{test} \\ \text{error}}}]$$

- Midterm analogy:
 - You have 2 practice midterms.
 - You hide one midterm, and spend a lot of time working through the other.
 - You then do the other practice term, to see how well you'll do on the test.
- We typically use validation error to choose “hyper-parameters”...

$$X_i = \begin{bmatrix} dx_i \\ \dots \\ \dots \\ \dots \end{bmatrix}^T$$

Coming Up Next

HYPER-PARAMETERS

Notation: Parameters and Hyper-Parameters

- The decision tree **rule** values are called “**parameters**”. $\{j, t, \hat{y}_{yes}, \hat{y}_{no}\}$
 - **Parameters control how well we fit** a dataset.
 - We “train” a model by trying to find the best parameters on training data.
 - The decision tree **depth** is called a “**hyper-parameter**”.
 - **Hyper-parameters control how complex our model is.**
 - We **can't “train” a hyper-parameter.**
 - You can always fit training data better by making the model more complicated.
 - We “validate” a hyper-parameter using a validation score.
 - (“Hyper-parameter” is sometimes used for parameters “not fit with data”.)
- Handwritten note: An arrow points from the word "depth" to the \hat{y}_{no} term in the set notation above.*

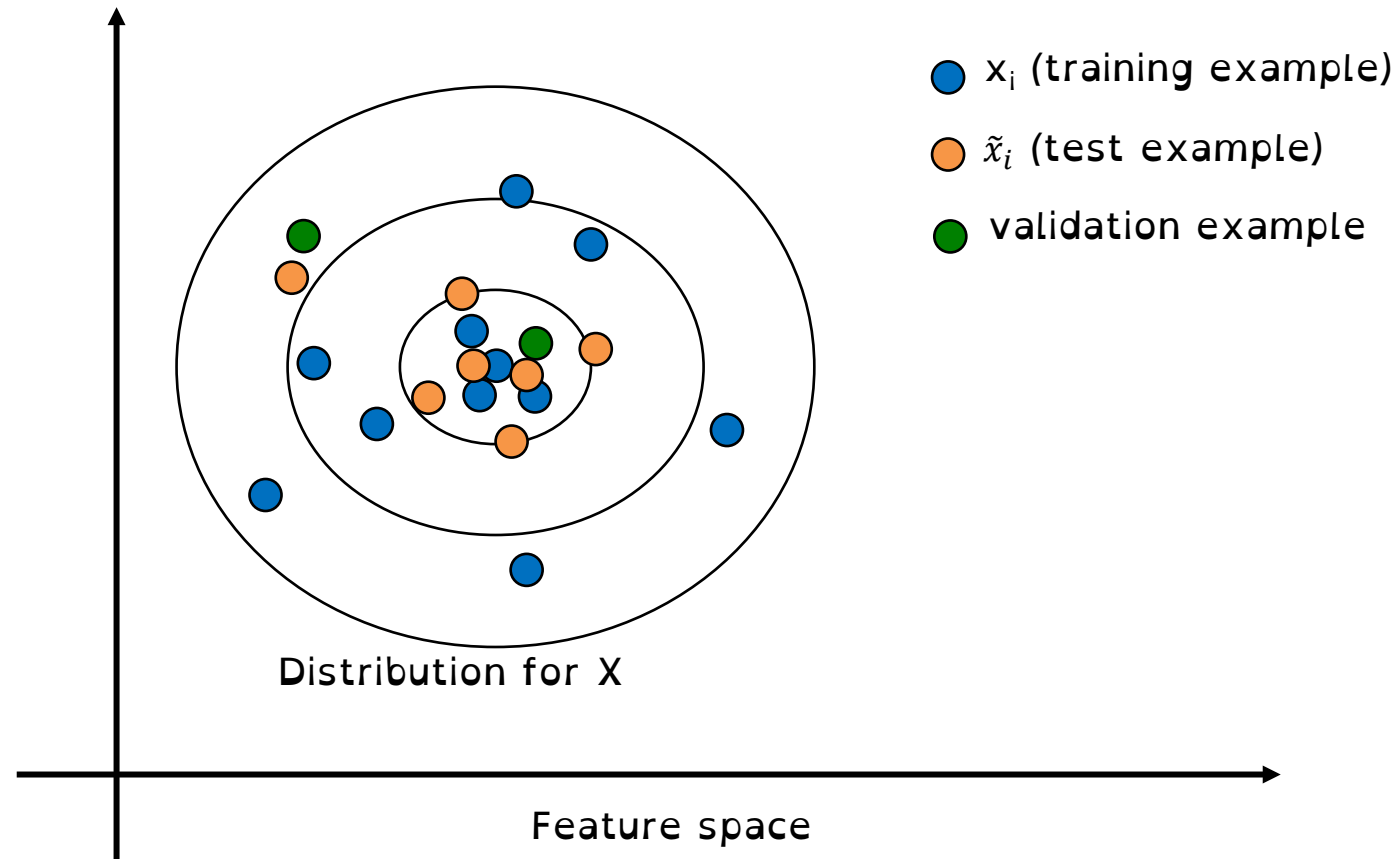
Choosing Hyper-Parameters with Validation Set

- So to choose a good value of depth (“hyper-parameter”), we could:
 - Try a depth-1 decision tree, compute validation error.
 - Try a depth-2 decision tree, compute validation error.
 - Try a depth-3 decision tree, compute validation error.
 - ...
 - Try a depth-20 decision tree, compute validation error.
 - Return the depth with the lowest validation error.
- After you choose the hyper-parameter, we usually re-train on the full training set with the chosen hyper-parameter.

Digression: Optimization Bias

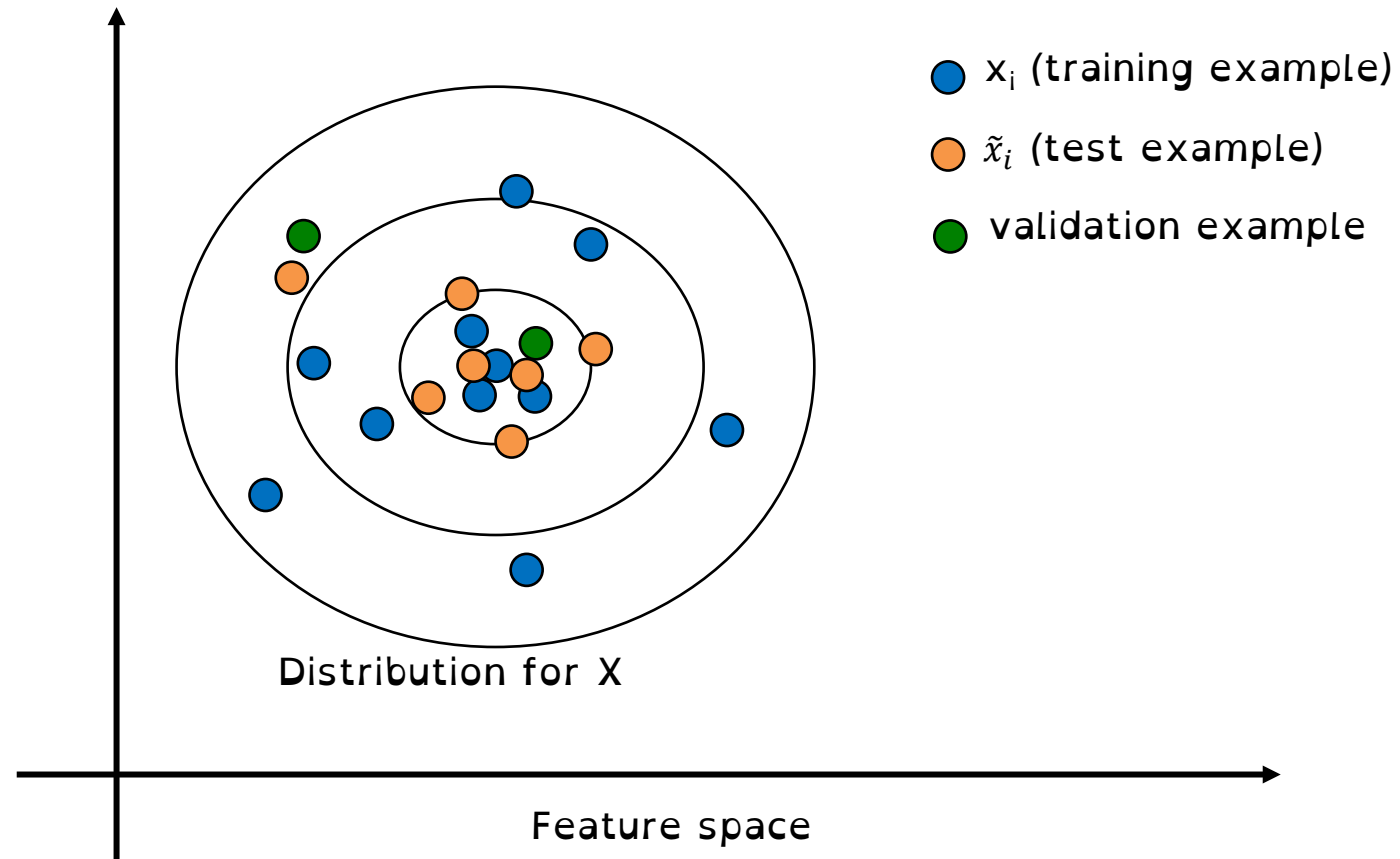
- Another name for overfitting is “**optimization bias**”:
 - How biased is an “error” that we optimized over many possibilities?
- **Optimization bias of parameter learning**:
 - During learning, we could search over tons of different decision trees.
 - So we can get “lucky” and find one with low training error **by chance**.
 - “Overfitting of the training error”.
- **Optimization bias of hyper-parameter tuning**:
 - Here, we might optimize the validation error over 20 values of “depth”.
 - One of the 20 trees might have low validation error **by chance**.
 - “Overfitting of the validation error”.

Visualizing Optimization Bias



I turned 2 training examples into validation examples.
Q: Does doing well on validation set mean we're going to do well on test set?

Visualizing Optimization Bias



- Say we looked at **1 million** different hyper-parameter values here
- Turns out I get a **0 validation error on the 2 validation examples** with some value
- It's **very unlikely** that this value will give me low test error though!

Digression: Example of Optimization Bias

- Consider a multiple-choice (a,b,c,d) “test” with 10 questions:
 - If you **choose answers randomly**, expected grade is 25% (no bias).
 - If you **fill out two tests randomly and pick the best**, expected grade is 33%.
 - Optimization bias of ~8%.
 - If you take the **best among 10** random tests, expected grade is ~47%.
 - If you take the **best among 100**, expected grade is ~62%.
 - If you take the **best among 1000**, expected grade is ~73%.
 - If you take the **best among 10000**, expected grade is ~82%.
 - You have so many “chances” that you expect to do well.
- But on **new questions the “random choice” accuracy is still 25%**.

Factors Affecting Optimization Bias

- If we instead used a **100-question test** then:
 - Expected grade from best over 1 randomly-filled test is 25%.
 - Expected grade from best over 2 randomly-filled test is ~27%.
 - Expected grade from best over 10 randomly-filled test is ~32%.
 - Expected grade from best over 100 randomly-filled test is ~36%.
 - Expected grade from best over 1000 randomly-filled test is ~40%.
 - Expected grade from best over 10000 randomly-filled test is ~47%.
- The **optimization bias grows with the number of things we try**.
 - “Complexity” of the set of models we search over.
- But, **optimization bias shrinks quickly with the number of examples**.
 - But it’s **still non-zero and growing** if you over-use your validation set!

Summary

- **Training error vs. testing error:**
 - What we care about in machine learning is the testing error.
- **Golden rule of machine learning:**
 - The test data cannot influence training the model in any way.
- **Independent and identically distributed (IID):**
 - One assumption that makes learning possible.
- **Fundamental trade-off:**
 - Trade-off between getting low training error and having training error approximate test error.
- **Validation set:**
 - We can save part of our training data to approximate test error.
- **Hyper-parameters:**
 - Parameters that control model complexity, typically set with a validation set.
- **Next time:**
 - We discuss the “best” machine learning method.

Review Questions

- Q1: What is the difference between memorization and Learning?
- Q2: In the SF/NY apartment example, why shouldn't I visualize the 217 new homes along with my training data?
- Q3: How do E_{approx} and E_{train} generally behave with respect to the complexity (aka expressivity) of a model?
- Q4: If validation error approximates test error, why don't we turn most of the training examples into validation examples?

More Discussion of Optimization Bias

? question ★

stop following

91 views

Where does the term optimization bias come from

Could you please explain where the name optimization bias comes from, and what does it mean when saying "something" is due to optimization bias.

I learned it is used interchangeably with over-fitting, but they are slightly different. In my opinion, overfitting is a result, a result of using overly complicated models. However, it is confusing when optimization bias used as a reason like "we can't find the true model because of optimization bias".

i Mark Schmidt 1 month ago

To me, optimization bias is a very-general abstract concept, and overfitting is a special case of optimization bias.

When we minimize the value of a statistic (like the test error, or a p-value, or whatever) for a particular dataset in terms of some possible "parameters", we expect the value of that statistic to be lower on that particular dataset than it would be on a new dataset. The optimization bias is how much lower we expect it be.

In we define overfitting as the difference between the training error and the test error, then overfitting is the optimization bias due to fitting the parameters on our particular dataset.

“Test Set” vs. “Test Error”

- Formally, the “test error” is the expected error of our model:

$$E[|\hat{y}_i - \tilde{y}_i|]$$

- Here I’m using absolute error between predictions and true labels.
 - But you could use squared error or other losses.
 - The expectation is taken over distribution of test examples.
 - Think of this as the “error with infinite data”.
 - We assume that our training examples are drawn IID from this distribution.
 - Otherwise, “training” might not help to reduce “test error”.
- Unfortunately, we cannot compute the test error.
 - We don’t have access to the distribution over all test examples.

“Test Set” vs. “Test Error”

- We often approximate “test error” with the error on a “test set”:

$$\frac{1}{t} \sum_{i=1}^t |\hat{y}^i - \tilde{y}^i|$$

- Here, we are using ‘t’ examples drawn IID from the test distribution.
- Note that “test set error” is not the “test error”.
 - The goal is have a low “test error”, not “test set error”.
- The “golden rule” of machine learning:
 - A “test set” cannot influence the “training” in any way.
 - Otherwise, “test set error” is not an unbiased “test error” approximation.
 - We run the risk of “overfitting” to the “test set”.

“test error” vs. “test set error” vs. “validation error”



Chenliang Zhou ✓✓ 8 months ago @Mark

About Q1, wouldn't the dataset we use to examine our performance be called validation dataset? Mike said that in 340 “testing dataset” refers to those we don't know.



Lucas Porto ✓ 8 months ago I'm now confused about this too. I thought there should be a separate “test set”, which you use to measure the performance of your model after training and selection. Selection here meaning hyperparameter tuning with a validation set that not used for training.



i Mark Schmidt 8 months ago Unfortunately, there isn't a standard nomenclature for what exactly defines a “test set”. But a common convention is this:

1. The “test error” is the expected error over all possible future examples. You can never measure this.
2. We often have a “test set” that we are using to approximate this “test error”. So we could say the “test set error” is being used as an approximation of the “test error”. If you want this “test set error” to be an unbiased estimate of test error, it should not influence the training in any way. Unfortunately, most people (including your profs) aren't careful about distinguishing “test error” and “test set error”.
3. When we tune hyper-parameters, we often use a “validation set” to approximate the “test error”. Since we are evaluating the validation error several times, it will have an optimization bias. So it might guide us towards good hyper-parameters (because the bias is typically not that large) but really be used as an unbiased measure of test error.

↳ can't

“test error” vs. “test set error” vs. “validation error”



We are given a huge dataset that we want to make a model from it. We can never know the exact performance of the model for new data that is NOT part of our dataset.

So here is what we can do:

Split the huge dataset into 3 categories:

- a) **Training data**: this data is used to train a model
- b) **Validation data**: this data is used "intermediate" measure the performance of the model we created
- c) **Test data**: this data is used as a "final" measure of performance of the of the final model that was created

To choose a model do the following:

1. train the model using the **training data**
2. once you have a candidate model, find its performance (i.e validation error) using the **validation data**
3. if you are not satisfied with the performance of the candidate model, find a new model using **training data** and measure its performance using **validation data**. But don't look at your **test data** yet. (i.e do step 1 and 2 again)
4. once you have a model that you are satisfied with (i.e it has low validation error) you can select the model as your **FINAL trained model** meaning that you cannot go back and change the model again.
5. Measure the performance of your **FINAL** model using the **Test data**. You can think of this performance, as the good approximation of the performance of the model on **NEW** data that the model has never seen.

And the golden rule of ML states that

You should NEVER EVER use your test data in order to train a model.

If you do so your model will be biased.



Mark Schmidt 8 months ago Great explanation anonymous!

“A visual Introduction to machine Learning”

- The “housing prices” example is taken from this website:
 - <http://www.r2d3.us/visual-intro-to-machine-learning-part-1>
- They also have a “Part 2” here:
 - <http://www.r2d3.us/visual-intro-to-machine-learning-part-2>
- Part 2 covers similar topics to what we covered in this lecture.

Approximation Error for Selecting Hyper-Parameters

- From the [2019 EasyMarkit AI Hackathon](#):
 - “We ended up selecting the hyperparameters that gave us the lowest approximation error (gap between train and validation) as opposed to the lowest validation error. This was quite a difficult decision for our team since we were only allowed one submission. However, the model with the lowest validation error had a very high approximation error, which felt too risky, so we went with a model with a slightly higher validation error and much lower approximation error. When the results were announced, the reported test accuracy was within 0.1% of what our model predicted with the validation set.”
- This is the type of reasoning you want to do.
 - A high approximation error could indicate low validation error by chance.

Typical Supervised Learning Steps (Are Bad?)

- Given data $\{X,y\}$, a typical set of supervised learning steps:
 - Data splitting:
 - Split $\{X,y\}$ into a train set $\{X_{train},y_{train}\}$ and a validation set $\{X_{valid},y_{valid}\}$.
 - We're going to use the validation set error as an approximation of test error.
 - Tune hyper-parameters (decision tree depth, “regularization”, “number of hidden units”, etc.):
 - For each candidate value “ λ ” of the hyper-parameters:
 - Fit a model to the train set $\{X_{train},y_{train}\}$ using the given hyper-parameters “ λ ”.
 - Evaluate the model on the validation set $\{X_{valid},y_{valid}\}$.
 - Choose the model with the best performance on the validation set.
 - And maybe re-train using hyper-parameter “ λ ” on the full dataset.
- Can this overfit, even though we used a validation set?
 - Yes, we've violated the golden rule. But maybe it's not too bad...

Validation Error, Test Error, and Approximation Error

- We discuss “**fundamental trade-off**” with respect to train error.
 - Simple identity relating training set error to test error.
- We have a **similar identity for the validation error**.
 - If E_{test} is the test error and E_{valid} is the error on the validation set, then:

$$E_{\text{test}} = \underbrace{(E_{\text{test}} - E_{\text{valid}})}_{E_{\text{approx}}} + E_{\text{valid}}$$

- If E_{approx} is small, then E_{valid} is a good approximation of E_{test} .
 - We can't measure E_{test} , so how do we know if E_{approx} is small?

Bounding E_{approx}

- Let's consider a simple case:
 - Labels y^i are binary, and we try 1 hyper-parameter setting.
 - IID assumption on validation set implies E_{valid} is unbiased: $E[E_{\text{valid}}] = E_{\text{test}}$.
- We can bound probability E_{approx} is greater than ϵ .
 - Assumptions: data is IID (so E_{valid} is unbiased) and loss is in $[0,1]$.
 - By using Hoeffding's inequality:

$$P(\underbrace{|E_{\text{test}} - E_{\text{valid}}|}_{E_{\text{approx}}} > \epsilon) \leq 2 \exp(-2 \epsilon^2 t)$$

↑ number of examples
in validation set

- Probability that E_{valid} is far from E_{test} goes down exponentially with 't'.
 - This is great: the bigger your validation set, the better approximation you get.

Bounding E_{approx}

- Let's consider a slightly less-simple case:
 - Labels are binary, and we tried '**k**' hyper-parameter values.
 - In this case it's unbiased for each '**k**': $E[E_{\text{valid}(\lambda)}] = E_{\text{test}}$.
 - So for each validation error $E_{\text{valid}(\lambda)}$ we have:

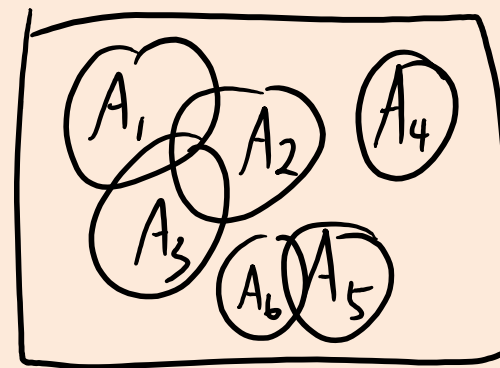
$$P(|E_{\text{test}} - E_{\text{valid}(\lambda)}| > \epsilon) \leq 2 \exp(-2 \epsilon^2 t)$$

- But our final validation error is $E_{\text{valid}} = \min\{E_{\text{valid}(\lambda)}\}$, which is **biased**.
 - We can't apply Hoeffding because we **chose best among 'k' values**.
- Fix: **bound on probability that all $|E_{\text{test}} - E_{\text{valid}(\lambda)}|$ values are $\leq \epsilon$** .
 - We show it holds for all values of λ , so it must hold for the best value.

Bounding E_{approx}

- The “**union bound**” for any events $\{A_1, A_2, \dots, A_k\}$ is that:

$$p(A_1 \cup A_2 \cup \dots \cup A_k) \leq \sum_{i=1}^k p(A_i)$$



Union of areas is less than sum of areas.

- Combining with Hoeffding we can get:

$$\begin{aligned} p(|E_{\text{test}} - \min_{\lambda} \{E_{\text{valid}(\lambda)}\}| > \epsilon) &\leq p(\text{Exists a } \lambda \text{ where } |E_{\text{test}} - E_{\text{valid}(\lambda)}| > \epsilon) \\ &\leq \sum_{\lambda} p(|E_{\text{test}} - E_{\text{valid}(\lambda)}| > \epsilon) \\ &\leq \sum_{\lambda} 2 \exp(-2\epsilon^2 t) \\ &= k 2 \exp(-2\epsilon^2 t) \end{aligned}$$

Bounding E_{approx}

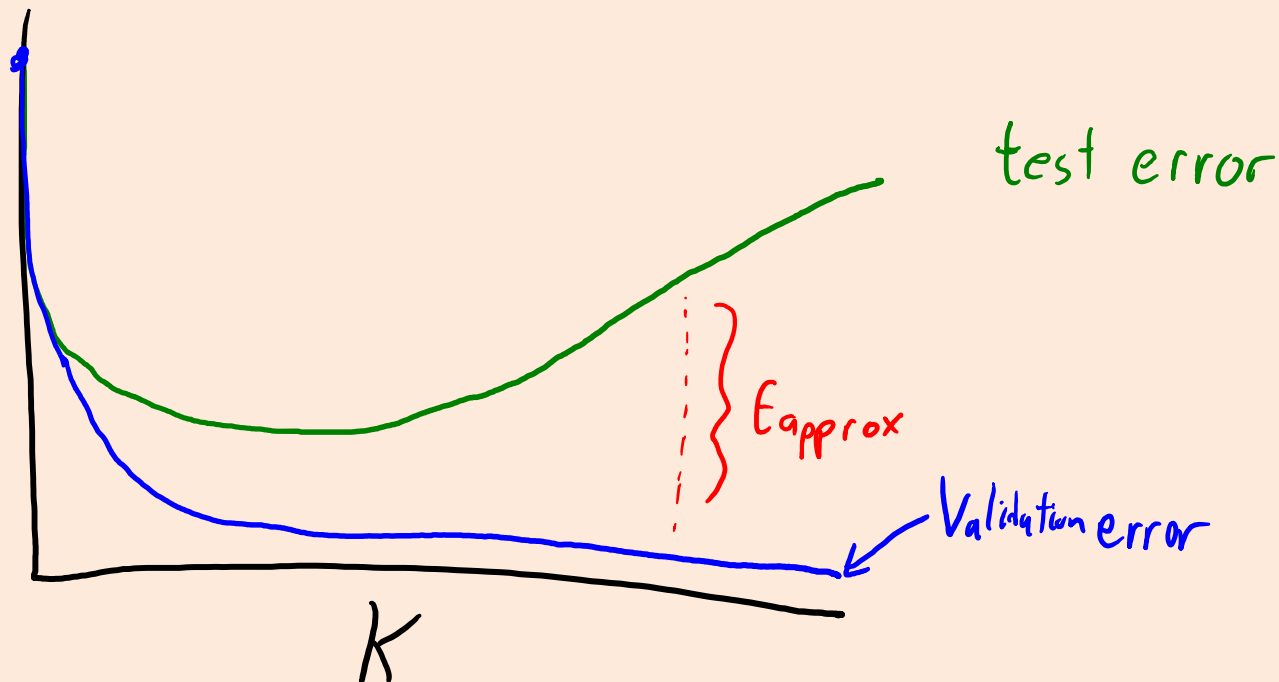
- So if we choose best $E_{\text{valid}(\lambda)}$ among 'k' λ values, we have:

$$p(|E_{\text{test}} - E_{\text{valid}(\lambda)}| > \epsilon \text{ for any } \lambda) \leq k 2 \exp(-2 \epsilon^2 t)$$

- So **optimizing over 'k' models is ok if we have a large 't'**.
 - But if 'k' is too large or 't' is too small the validation error isn't useful.
- **Examples:**
 - If $k=10$ and $t=1000$, probability that $|E_{\text{approx}}| > .05$ is less than 0.14.
 - If $k=10$ and $t=10000$, probability that $|E_{\text{approx}}| > .05$ is less than 10^{-20} .
 - If $k=10$ and $t=1000$, probability that $|E_{\text{approx}}| > .01$ is less than 2.7 (useless).
 - If $k=100$ and $t=100000$, probability that $|E_{\text{approx}}| > .01$ is less than 10^{-6} .

Bounding E_{approx}

- Validation error vs. test error for fixed 't'.
 - E_{valid} goes down as we increase 'k', but E_{approx} can go up.
 - **Overfitting** of validation set.



Discussion

- Bound is usually very loose, but data is probably not fully IID.
 - Similar bounds are possible for cross-validation.
- Similar arguments apply for the E_{approx} of the training error.
 - Value 'k' is the number of hyper-parameters you are optimizing over (even if don't try them all).
 - So 'k' is usually huge: you try out $k=O(nd)$ decision stumps.
- What if we train by optimizing parameters over a continuous space?
 - We're optimizing on continuous space, so $k=\infty$ and the bound is useless.
 - In this case, VC-dimension is one way to replace 'k' (doesn't need union bound).
 - "Simpler" models like decision stumps and linear models will have lower VC-dimension.
- Learning theory keywords if you want to go deeper into this topic:
 - Bias-variance (see bonus slides for details and why this is weird), sample complexity, PAC learning, VC dimension, Rademacher complexity.
 - A gentle place to start is the [Learning from Data book](#).

Refined Fundamental Trade-Off

- Let E_{best} be the **irreducible error** (lowest possible error for any model).
 - For example, irreducible error for predicting coin flips is 0.5.
- Some learning theory results use E_{best} to further decompose E_{test} :

$$E_{\text{test}} = \underbrace{(E_{\text{test}} - E_{\text{train}})}_{E_{\text{approx}}} + \underbrace{(E_{\text{train}} - E_{\text{best}})}_{E_{\text{model}}} + \underbrace{E_{\text{best}}}_{\text{"noise"}}$$

- This is similar to the bias-variance trade-off:
 - E_{approx} measures how sensitive we are to training data (like “variance”).
 - E_{model} measures if our model is complicated enough to fit data (like “bias”).
 - E_{best} measures how low can any model make test error (“irreducible” error).

Refined Fundamental Trade-Off

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- This is similar to the bias-variance trade-off:
 - You need to trade between having low E_{approx} and having low E_{model} .
 - Powerful models have low E_{model} but can have high E_{approx} .
 - E_{best} does not depend on what model you choose.

Bias-Variance Decomposition

- You may have seen “**bias-variance decomposition**” in other classes:
 - Assumes $\tilde{y}_i = \bar{y}_i + \varepsilon$, where ε has mean 0 and variance σ^2 .
 - Assumes we have a “learner” that can take ‘n’ training examples and use these to make predictions \hat{y}_i .

- **Expected squared test error** in this setting is

$$\mathbb{E} \left[(\tilde{y}_i - \hat{y}_i)^2 \right] = \underbrace{\mathbb{E} \left[(\hat{y}_i - \bar{y}_i)^2 \right]}_{\text{"bias"}} + \underbrace{\left(\mathbb{E} \left[\hat{y}_i^2 \right] - \mathbb{E} \left[\hat{y}_i \right]^2 \right)}_{\text{"variance"}} + \underbrace{\sigma^2}_{\text{"noise"}}$$

Note: The term $\mathbb{E} \left[(\tilde{y}_i - \hat{y}_i)^2 \right]$ is labeled "test squared error" in the original image.

- Where **expectations are taken over possible training sets** of ‘n’ examples.
- **Bias** is expected error due to having wrong model.
- **Variance** is expected error due to sensitivity to the training set.
- **Noise** (irreducible error) is the best we can hope for given the noise (E_{best}).

Refined Fundamental Trade-Off

- Decision tree with **high depth**:
 - Very likely to fit data well, so **bias is low**.
 - But model changes a lot if you change the data, so **variance is high**.
- Decision tree with **low depth**:
 - Less likely to fit data well, so **bias is high**.
 - But model doesn't change much you change data, so **variance is low**.
- And **degree does not affect irreducible error**.
 - Irreducible error comes from the best possible model.

Bias-Variance vs. Fundamental Trade-Off

- Both decompositions **serve the same purpose:**
 - Trying to evaluate how different factors affect test error.
- They both lead to the same 3 conclusions:
 1. Simple models can have high E_{train} /bias, low E_{approx} /variance.
 2. Complex models can have low E_{train} /bias, high E_{approx} /variance.
 3. As you increase 'n', E_{approx} /variance goes down (for fixed complexity).

Bias-Variance vs. Fundamental Trade-Off

- So why focus on fundamental trade-off and not bias-variance?
 - **Simplest** viewpoint that gives these 3 conclusions.
 - **No assumptions** like being restricted to squared error.
 - You can **measure E_{train}** but **not E_{approx}** (1 known and 1 unknown).
 - If E_{train} is low and you expect E_{approx} to be low, then you are happy.
 - E.g., you fit a very simple model or you used a huge independent validation set.
 - You **can't measure bias, variance, or noise** (3 unknowns).
 - If E_{train} is low, bias-variance decomposition doesn't say anything about test error.
 - You only have your training set, not distribution over possible datasets.
 - Doesn't say if high E_{test} is due to bias or variance or noise.

Learning Theory

- Bias-variance decomposition is a bit weird:
 - Considers expectation over *possible training sets*.
- Bias-variance says *nothing about your training set*.
 - This is different than Hoeffding bounds:
 - Bound the test error based on your actual training set and training/validation error.
- Some keywords if you want to learn about learning theory:
 - Bias-variance decomposition, sample complexity, probably approximately correct (PAC) learning, Vapnik-Chernovenkis (VC) dimension, Rademacher complexity.
- A gentle place to start is the “Learning from Data” book:
 - <https://work.caltech.edu/telecourse.html>

A Theoretical Answer to “How Much Data?”

- Assume we have a source of IID examples and a fixed class of parametric models.
 - Like “all depth-5 decision trees”.
- Under some nasty assumptions, with ‘n’ training examples it holds that:
 $E[\text{test error of best model on training set}] - (\text{best test error in class}) = O(1/n)$.
- You rarely know the constant factor, but this gives some guidelines:
 - Adding more data helps more on small datasets than on large datasets.
 - Going from 10 training examples to 20, difference with best possible error gets cut in half.
 - If the best possible error is 15% you might go from 20% to 17.5% (this does not mean 20% to 10%).
 - Going from 110 training examples to 120, error only goes down by ~10%.
 - Going from 1M training examples to 1M+10, you won’t notice a change.
 - Doubling the data size cuts the error in half:
 - Going from 1M training to 2M training examples, error gets cut in half.
 - If you double the data size and your test error doesn’t improve, more data might not help.

Can you test the IID assumption?

- In general, **testing the IID assumption** is not easy.
 - Usually, you need background knowledge to decide if it's reasonable.
- Some tests do exist, like shuffling the order of data and then measuring if some basic statistics agree.
 - It's reasonable to check if summary statistics of train and test data agree.
 - If not, your trained model may not be so useful.
- Some discussion here:
 - <https://stats.stackexchange.com/questions/28715/test-for-iid-sampling>

Wrong Decisions under false IID Assumption

There is a different narrative that one can tell about the current era. Consider the following story, which involves humans, computers, data, and life-or-death decisions, but where the focus is something other than intelligence-in-silicon fantasies. When my spouse was pregnant 14 years ago, we had an ultrasound. There was a geneticist in the room, and she pointed out some white spots around the heart of the fetus. “Those are markers for Down syndrome,” she noted, “and your risk has now gone up to one in 20.” She let us know that we could learn whether the fetus in fact had the genetic modification underlying Down syndrome via an amniocentesis, but amniocentesis was risky—the chance of killing the fetus during the procedure was roughly one in 300. Being a statistician, I was determined to find out where these numbers were coming from. In my research, I discovered that a statistical analysis had been done a decade previously in the UK in which these white spots, which reflect calcium buildup, were indeed established as a predictor of Down syndrome. I also noticed that the imaging machine used in our test had a few hundred more pixels per square inch than the machine used in the UK study. I returned to tell the geneticist that I believed that the white spots were likely false positives, literal white noise.

She said, “Ah, that explains why we started seeing an uptick in Down syndrome diagnoses a few years ago. That’s when the new machine arrived.”

We didn’t do the amniocentesis, and my wife delivered a healthy girl a few months later, but the episode troubled me, particularly after a back-of-the-envelope calculation convinced me that many thousands of people had gotten that diagnosis that same day worldwide, that many of them had opted for amniocentesis, and that a number of babies had died needlessly. The problem that this episode revealed wasn’t about my individual medical care; it was about a medical system that measured variables and outcomes in various places and times, conducted statistical analyses, and made use of the results in other situations. The problem had to do not just with data analysis per se, but with what database researchers call *provenance*—broadly, where did data arise, what inferences were drawn from the data, and how relevant are those inferences to the present situation? While a trained human might be able to work all of this out on a case-by-case basis, the issue was that of designing a planetary-scale medical system that could do this without the need for such detailed human oversight.

I’m also a computer scientist, and it occurred to me that the principles needed to build planetary-scale inference-and-decision-making systems of this kind, blending computer science with statistics, and considering human utilities, were nowhere to be found in my education. It occurred to me that the development of such principles—which will be needed not only in the medical domain but also in domains such as commerce, transportation, and education—were at least as important as those of building AI systems that can dazzle us with their game-playing or sensorimotor skills.