Notes

- Final Project
  - Please contact me this week with ideas, so we can work out a good topic

Reduced Coordinates

- Constraint methods from last class involved adding forces, variables etc. to remove degrees of freedom
- Inevitably have to deal with drift, error, …
- Instead can (sometimes) formulate problem to directly eliminate degrees of freedom
  - Give up some flexibility in exchange for eliminating drift, possibly running a lot faster
- "Holonomic constraints": if we have n true degrees of freedom, can express current position of system with n variables
  - Rigid bodies: centre of mass and Euler angles
  - Articulated rigid bodies: base link and joint angles

Finding the equations of motion

- Unconstrained system state is x, but holonomic constraints mean x=x(q)
  - The vector q is the "generalized" or "reduced" coordinates of the system
  - dim(q) < dim(x)
- Suppose our unconstrained dynamics are

$$\tfrac{d}{dt}\left(Mv\right) = F$$

  - Could include rigid bodies if M includes inertia tensors as well as standard mass matrices
- What will the dynamics be in terms of q?

Principle of virtual work

- Differentiate x=x(q): $v = \frac{\partial x}{\partial q}\dot{q}$
- That is, legal velocities are some linear combination of the columns of $\frac{\partial x}{\partial q}$
  - (coefficients of that combination are just dq/dt)
- Principle of virtual work: constraint force must be orthogonal to this space

$$\frac{\partial x}{\partial q}^T F_{\text{constraint}} = 0$$

Equation of motion

- Putting it together, just like rigid bodies,

$$\frac{\partial x}{\partial q}^T \frac{\partial}{\partial t}\left(M \frac{\partial x}{\partial q}\dot{q}\right) = \frac{\partial x}{\partial q}^T F$$

$$\left(\frac{\partial x}{\partial q}^T M \frac{\partial x}{\partial q}\right)\ddot{q} + \frac{\partial x}{\partial q}^T \dot{M} \frac{\partial x}{\partial q}\dot{q} + \frac{\partial x}{\partial q}^T M \frac{\partial v}{\partial q}\dot{q} = \frac{\partial x}{\partial q}^T F$$

  - Note we get a matrix times second derivatives, which we can invert at any point for second order time integration
  - Generalized forces on right hand side
  - Other terms are pseudo-forces (e.g. Coriolis, centrifugal force, …)

Generalized Forces

- Sometimes the force is known on the system, and so the generalized force just needs to be calculated
  - E.g. gravity
- But often we don't care what the true force is, just what its effect is: directly specify the generalized forces
  - E.g. joint torques

Cleaning things up

- Equations are rather messy still
- Classical mechanics has spent a long time playing with the equations to make them nicer
  - And extend to include non-holonomic constraints for example
- Let's look at one of the traditional approaches: Lagrangian mechanics

Setting up Lagrangian Equations

- For simplicity, assume we model our system with N point masses, positions controlled by generalized coordinates
- We'll work out equations via kinetic energy
- As before $F_{\text{constraint}} + F = Ma$
- Using principle of virtual work, can eliminate constraint forces: $$\frac{\partial x}{\partial q}^T F = \frac{\partial x}{\partial q}^T Ma$$
- Equation j is just $$\sum_{i=1}^{N} \frac{\partial \vec{x}_i}{\partial q_j} \cdot \vec{F}_i = \sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{x}_i}{\partial q_j}$$

Introducing Kinetic Energy

$$\begin{aligned}
\sum_{i=1}^{N} m_i \vec{a}_i \cdot \frac{\partial \vec{x}_i}{\partial q_j} &= \sum_{i=1}^{N} m_i \left(\frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial \vec{x}_i}{\partial q_j} \right) - \vec{v}_i \cdot \frac{d}{dt}\frac{\partial \vec{x}_i}{\partial q_j} \right) \\
&= \sum_{i=1}^{N} m_i \left(\frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right) \\
&= \sum_{i=1}^{N} m_i \left(\frac{d}{dt}\left(\tfrac{1}{2} \frac{\partial}{\partial \dot{q}_j} |v_i|^2 \right) - \tfrac{1}{2} \frac{\partial}{\partial q_j} |v_i|^2 \right) \\
&= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \sum_{i=1}^{N} \tfrac{1}{2} m_i |v_i|^2 \right) - \frac{\partial}{\partial q_j} \sum_{i=1}^{N} \tfrac{1}{2} m_i |v_i|^2 \\
&= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}
\end{aligned}$$

Lagrangian Equations of Motion

- Label the j'th generalized force $$f_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q_j}$$
- Then the Lagrangian equations of motion are (for j=1, 2, …): $$f_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$$

Potential Forces

- If force on system is the negative gradient of a potential W (e.g. gravity, undamped springs, …) then further simplification: $$f_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q_j} = \sum_{i=1}^{N} -\frac{\partial W}{\partial \vec{x}_i} \frac{\partial \vec{x}_i}{\partial q_j} = -\frac{\partial W}{\partial q_j}$$
- Plugging this in: $$-\frac{\partial W}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial (T - W)}{\partial q_j}$$
- Defining the Lagrangian L=T-W, $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

Implementation

- For any kind of reasonably interesting articulated figure, expressions are truly horrific to work out by hand
- Use computer: symbolic computing, automatic differentiation
- Input a description of the figure
- Program outputs code that can evaluate terms of differential equation
- Use whatever numerical solver you want (e.g. Runge-Kutta)
- Need to invert matrix every time step in a numerical integrator
  - Gimbal lock…

Fluid mechanics

Fluid mechanics

- We already figured out the equations of motion for continuum mechanics $\rho \ddot{x} = \nabla \cdot \sigma + \rho g$
- Just need a constitutive model

$$\sigma = \sigma\left(x, t, \varepsilon, \dot{\varepsilon}\right)$$

- We'll look at the constitutive model for "Newtonian" fluids today
  - Remarkably good model for water, air, and many other simple fluids
  - Only starts to break down in extreme situations, or more complex fluids (e.g. viscoelastic substances)

Inviscid Euler model

- Inviscid=no viscosity
- Great model for most situations
  - Numerical methods end up with viscosity-like error terms anyways…
- Constitutive law is very simple: $\sigma_{ij} = -p\delta_{ij}$
  - New scalar unknown: pressure p
  - Barotropic flows: p is just a function of density (e.g. perfect gas law $p=k(\rho-\rho_0)+p_0$ perhaps)
  - For more complex flows need heavy-duty thermodynamics: an equation of state for pressure, equation for evolution of internal energy (heat), …

Lagrangian viewpoint

- We've been working with Lagrangian methods so far
  - Identify chunks of material,
    track their motion in time,
    differentiate world-space position or velocity w.r.t. material coordinates to get forces
  - In particular, use a mesh connecting particles to approximate derivatives (with FVM or FEM)
- Bad idea for most fluids
  - [vortices, turbulence]
  - At least with a fixed mesh…

Eulerian viewpoint

- Take a fixed grid in world space, track how velocity changes at a point
- Even for the craziest of flows, our grid is always nice
- (Usually) forget about object space and where a chunk of material originally came from
  - Irrelevant for extreme inelasticity
  - Just keep track of velocity, density, and whatever else is needed

Conservation laws

- Identify any fixed volume of space
- Integrate some conserved quantity in it (e.g. mass, momentum, energy, …)
- Integral changes in time only according to how fast it is being transferred from/to surrounding space $\frac{\partial}{\partial t}\int_\Omega q = -\int_{\partial\Omega} f(q) \cdot n$
  - Called the flux
  - [divergence form] $q_t + \nabla \cdot f = 0$

Conservation of Mass

- Also called the continuity equation (makes sure matter is continuous)
- Let's look at the total mass of a volume (integral of density)
- Mass can only be transferred by moving it: flux must be ρu

$$\frac{\partial}{\partial t}\int_{\Omega}\rho = -\int_{\partial\Omega}\rho u \cdot n$$

$$\rho_t + \nabla\cdot(\rho u) = 0$$

Material derivative

- A lot of physics just naturally happens in the Lagrangian viewpoint
  - E.g. the acceleration of a material point results from the sum of forces on it
  - How do we relate that to rate of change of velocity measured at a fixed point in space?
  - Can't directly: need to get at Lagrangian stuff somehow
- The material derivative of a property q of the material (i.e. a quantity that gets carried along with the fluid) is $\frac{Dq}{Dt}$

Finding the material derivative

- Using object-space coordinates p and map x=X(p) to world-space, then material derivative is just

$$\begin{aligned}\frac{D}{Dt}q(t,x) &= \frac{d}{dt}q\big(t,X(t,p)\big)\\ &= \frac{\partial q}{\partial t} + \nabla q\cdot\frac{\partial x}{\partial t}\\ &= q_t + u\cdot\nabla q\end{aligned}$$

- Notation: u is velocity (in fluids, usually use u but occasionally v or V, and components of the velocity vector are sometimes u,v,w)

Compressible Flow

- In general, density changes as fluid compresses or expands
- When is this important?
  - Sound waves (and/or high speed flow where motion is getting close to speed of sound - Mach numbers above 0.3?)
  - Shock waves
- Often not important scientifically, almost never visually significant
  - Though the **effect** of e.g. a blast wave is visible! But the shock dynamics usually can be hugely simplified for graphics

Incompressible flow

- So we'll just look at incompressible flow, where density of a chunk of fluid never changes
  - Note: fluid density may not be constant throughout space - different fluids mixed together…
- That is, $D\rho/Dt=0$

Simplifying

- Incompressibility: $\frac{D\rho}{Dt} = \rho_t + u\cdot\nabla\rho = 0$
- Conservation of mass: $\rho_t + \nabla\cdot(\rho u) = 0$

$$\rho_t + \nabla\rho\cdot u + \rho\nabla\cdot u = 0$$

- Subtract the two equations, divide by ρ:

$$\nabla\cdot u = 0$$

- Incompressible == divergence-free velocity
  - Even if density isn't uniform!

Conservation of momentum

- Short cut: in $\rho \ddot{x} = \nabla \cdot \sigma + \rho g$

  use material derivative:

$$\rho \frac{Du}{Dt} = \nabla \cdot \sigma + \rho g$$
$$\rho(u_t + u \cdot \nabla u) = \nabla \cdot \sigma + \rho g$$

- Or go by conservation law, with the flux due to transport of momentum and due to stress:
  - Equivalent, using conservation of mass

$$(\rho u)_t + \nabla \cdot (u \rho u - \sigma) = \rho g$$

Inviscid momentum equation

- Plug in simplest consitutive law (σ=-pδ) from before to get

$$\rho(u_t + u \cdot \nabla u) = -\nabla p + \rho g$$
$$u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g$$

  - Together with conservation of mass: the Euler equations

Incompressible inviscid flow

- So the equations are: $u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g$

$$\nabla \cdot u = 0$$

- 4 equations, 4 unknowns (u, p)
- Pressure p is just whatever it takes to make velocity divergence-free
- In fact, incompressibility is a hard constraint; div and grad are transposes of each other and pressure p is the Lagrange multiplier
  - Just like we figured out constraint forces before…

Pressure solve

- To see what pressure is, take divergence of momentum equation

$$\nabla \cdot \left(u_t + u \cdot \nabla u + \tfrac{1}{\rho} \nabla p - g\right) = 0$$
$$\nabla \cdot \left(\tfrac{1}{\rho} \nabla p\right) = -\nabla \cdot \left(u_t + u \cdot \nabla u - g\right)$$

- For constant density, just get Laplacian (and this is Poisson's equation)
- Important numerical methods use this approach to find pressure

Projection

- Note that $\nabla \cdot u_t = 0$ so in fact

$$\nabla \cdot \tfrac{1}{\rho} \nabla p = -\nabla \cdot \left(u \cdot \nabla u - g\right)$$

- After we add $\nabla p/\rho$ to $u \cdot \nabla u$, divergence must be zero
- So if we tried to solve for additional pressure, we get zero
- Pressure solve is linear too
- Thus what we're really doing is a **projection** of $u \cdot \nabla u - g$ onto the subspace of divergence-free functions: $u_t + P(u \cdot \nabla u - g) = 0$