Physics 153 Section T0H - Solution to Problem 5

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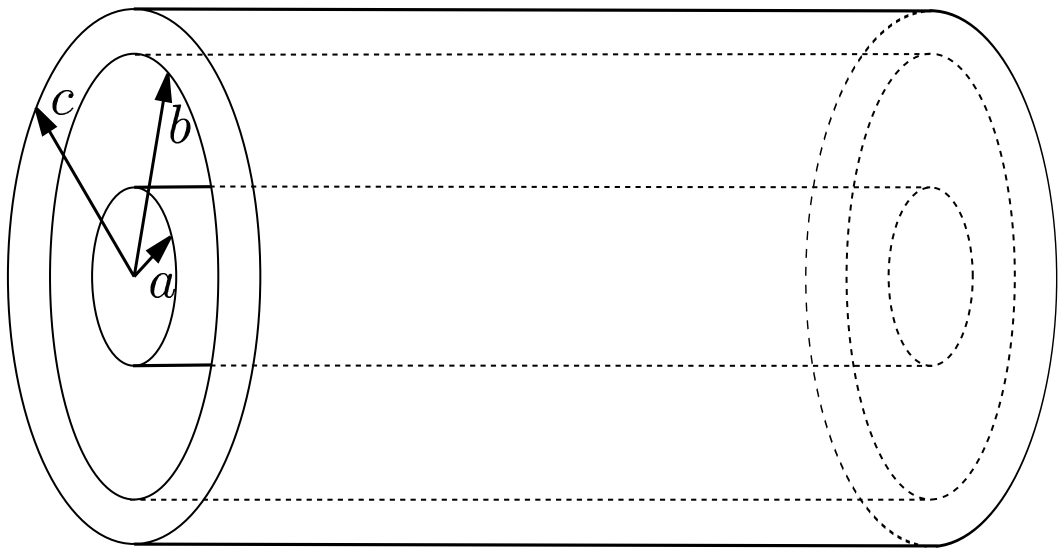
1 Assigned Problem



A very long, uniformly charged cylinder with volume charge density ρ and radius a is coaxial with a conducting cylindrical shell with inner radius b and outer radius c, as shown above. (a) Calculate and plot the electric field strength E as a function of radius r. (b) What is the surface charge density σ on the inner surface of the shell ($r = b$)?

2 Solution

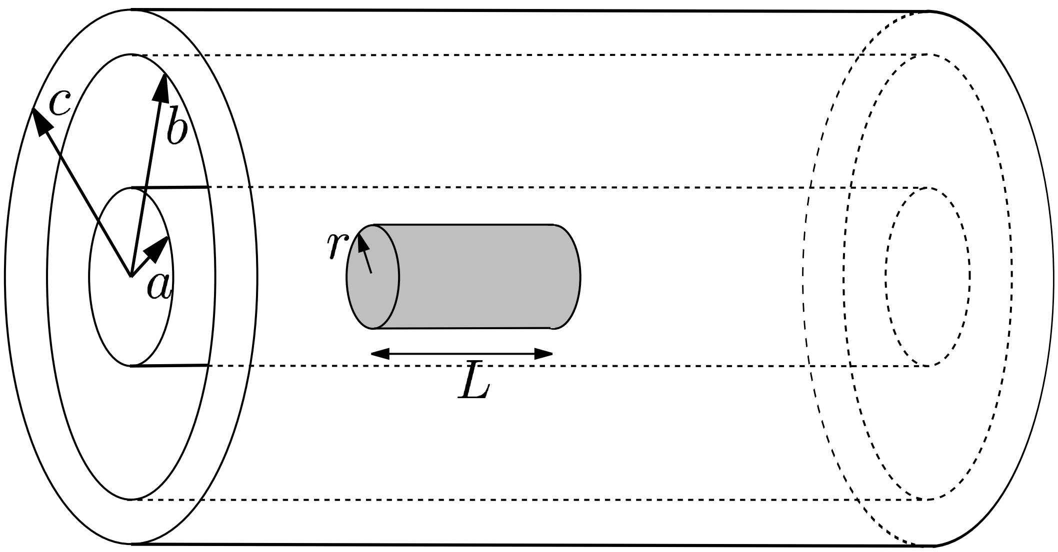
2.1 Part (a)

We have to use Gauss's law to calculate the electric field in each region: (1) $0 < r < a$, (2) $a < r < b$, (3) $b < r < c$, and (4) $r > c$.

To calculate the electric field we use Gauss's law. First we have to decide on a Gaussian surface. We interpret the phrase "very long" as practically *infinite*. That means we can apply some symmetries to simplify the problem. If the cylinders are infinite then the electric field lines must point radially away from the axis.

Now remember, constructing a Gaussian surface is easiest if you choose faces which are all parallel or perpendicular to the electric field. In this case, then, the easiest surface is another coaxial cylinder consisting of a face ($\mathbf{E} \cdot d\mathbf{A} = E\,dA$, parallel) and two ends ($\mathbf{E} \cdot d\mathbf{A} = 0$, perpendicular).

2.1.1 Region 1: $0 < r < a$



Here, the Gaussian surface is entirely within the charged cylinder so it encompasses a charge of

$$Q_{enc} = \rho \pi r^2 L. \tag{1}$$

By symmetry, the electric field lines must point radially away from the axis

$$\mathbf{E}(r) = E(r)\hat{\mathbf{r}} \tag{2}$$

so the flux is

$$\begin{aligned} \Phi &= \int \mathbf{E} \cdot d\mathbf{A} && (3)\\ &= \int_{\text{face}} E(r) dA + \int_{\text{ends}} 0 \, dA && (4)\\ &= E(r) \int_{\text{face}} dA && (5)\\ &= E(r) 2\pi r L && (6) \end{aligned}$$

because the electric field only contributes flux across the face of the Gaussian cylinder, not the ends.

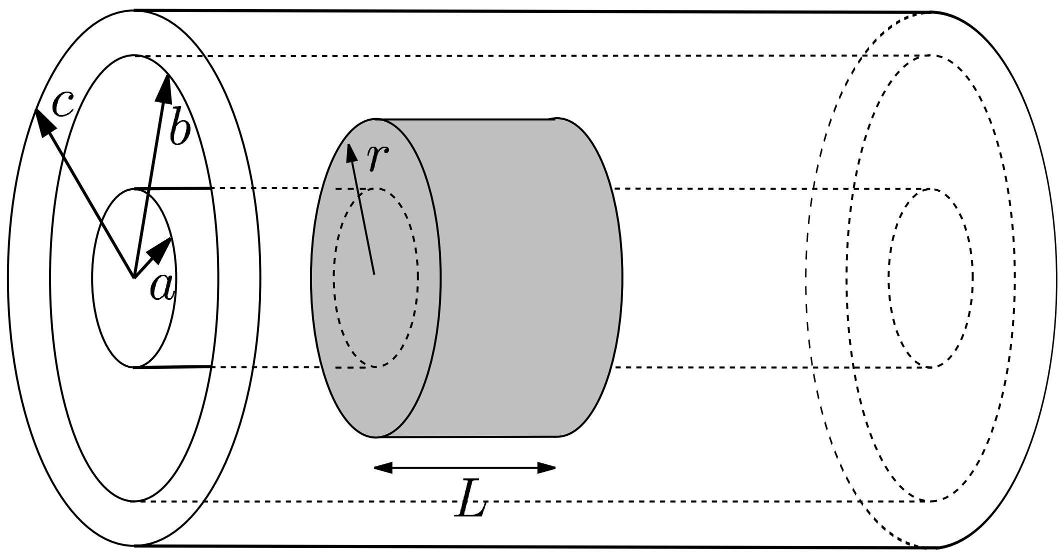
Now we apply Gauss's law, which says

$$\begin{aligned} \Phi &= Q_{enc}/\epsilon_0 && (7)\\ E(r)2\pi r L &= \rho \pi r^2 L/\epsilon_0 && (8) \end{aligned}$$

so

$$E(r) = \frac{\rho r}{2\epsilon_0}, \; 0 < r < a. \tag{9}$$

2.1.2 Region 2: $a < r < b$



In this region, the Gaussian surface is in empty space, completely enclosing the charged cylinder. Therefore, the enclosed charge is constant

$$Q_{enc} = \rho \pi a^2 L. \tag{10}$$

The flux still satisfies Eq. (6) so Gauss's law becomes

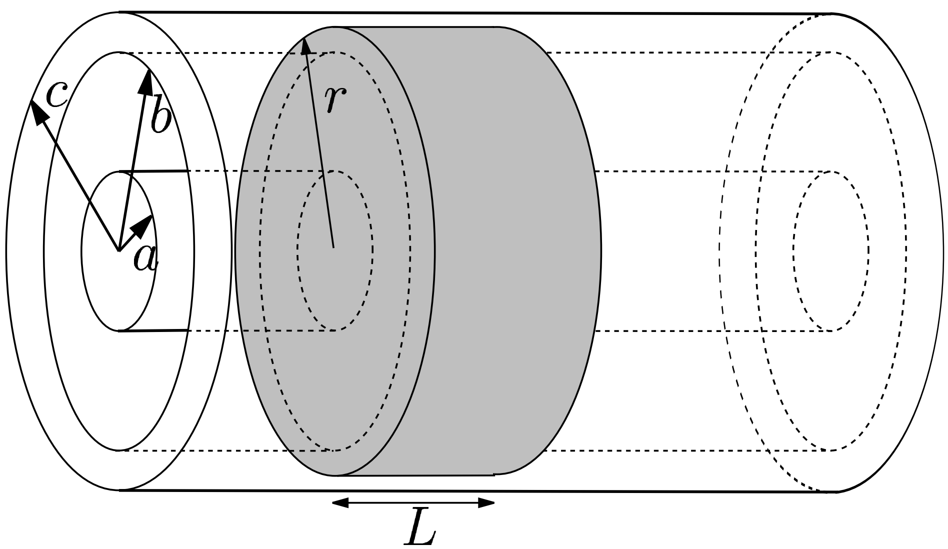
$$\Phi = Q_{enc}/\epsilon_0 \tag{11}$$

$$E(r) 2\pi r L \; = \; \rho \pi a^2 L / \epsilon_0 \tag{12}$$

and the electric field is

$$E(r) = \frac{\rho a^2}{2\epsilon_0 r}, \; a < r < b. \tag{13}$$

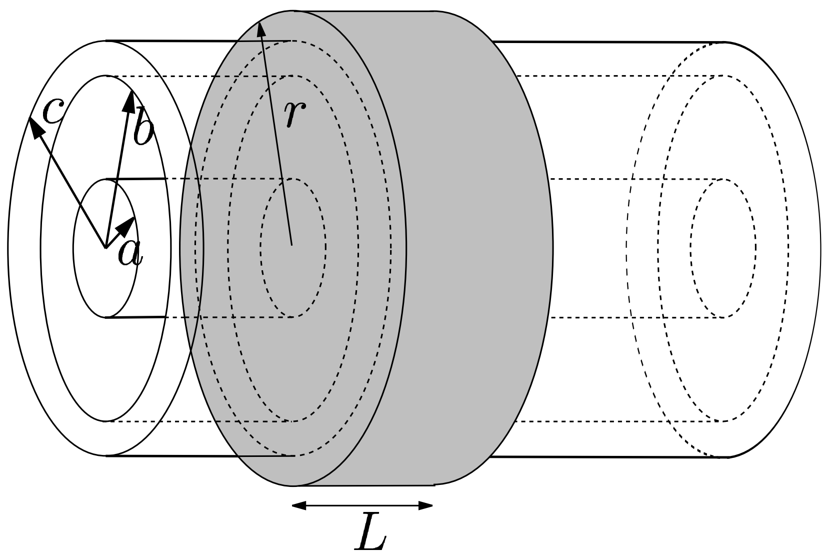
2.1.3 Region 3: $b < r < c$



This is the easiest region...the Gaussian surface is inside a conductor so the electric field at the surface must be zero

$$E(r) = 0,\; b < r < c. \tag{14}$$

2.1.4 Region 4: $r > c$



Now the Gaussian surface is outside of all the cylinders and, because the cylindrical shell has no charge, the enclosed charge is just given by Eq. (10) again.

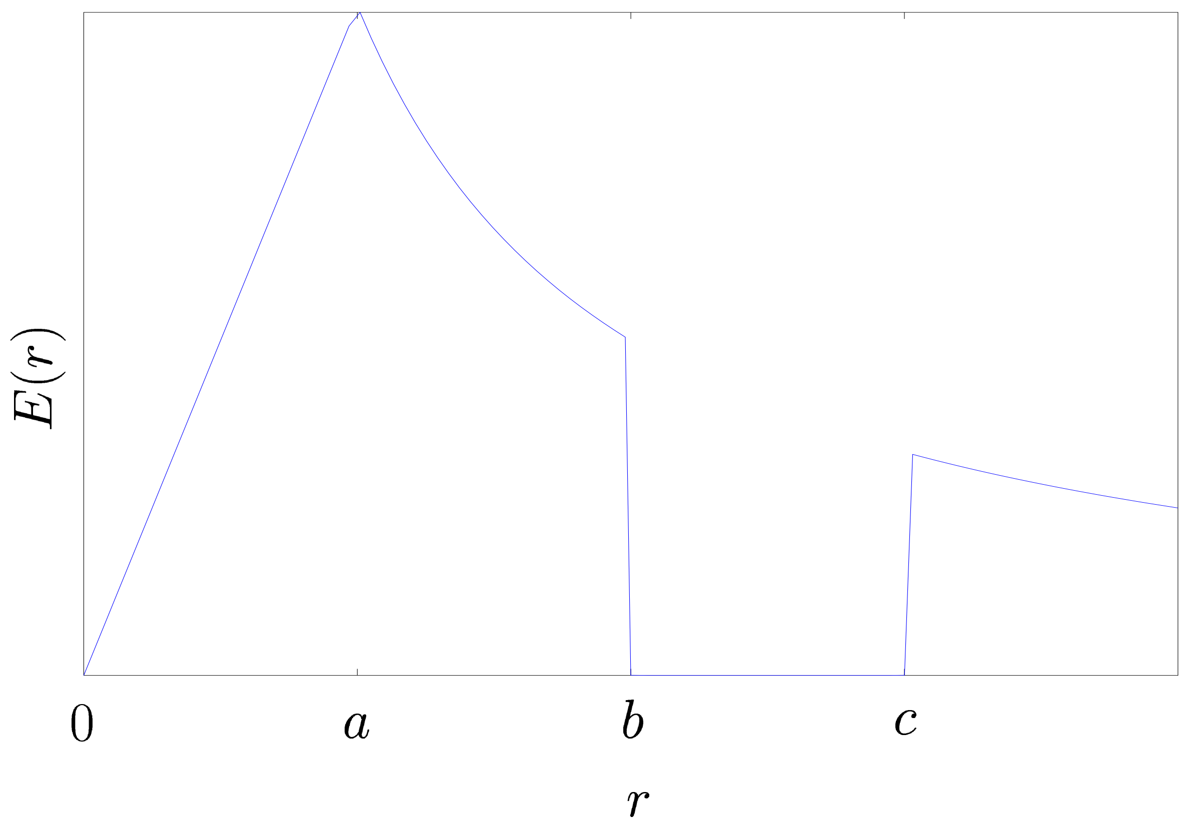
Also, the flux still satisfies Eq. (6) so Gauss's law becomes

$$\Phi = Q_{enc}/\epsilon_0 \tag{15}$$
$$E(r)2\pi r L = \rho\pi a^2 L/\epsilon_0 \tag{16}$$

and the electric field is

$$E(r) = \frac{\rho a^2}{2\epsilon_0 r},\ r > b. \tag{17}$$

Bringing together all four regions gives the following plot for the electric field strength as a function of radius:



2.2 Part (b)

We know from Gauss's law in Region 3 ($b < r < c$) that the total enclosed charge must be zero (because the electric field is zero, so the flux is zero...). Given the inner cylinder has a volume charge density ρ and the inner surface of the cylindrical shell has an unknown surface charge density σ

$$\begin{aligned} 0 &= Q_{enc} &(18)\\ &= \rho\pi a^2 L + \sigma 2\pi b L. &(19)\end{aligned}$$

Solving this immediately gives the charge density required to balance the inner cylinder

$$\sigma = -\frac{\rho a^2}{2b}. \tag{20}$$