Physics 153 Section T0H - Solution to Problem 6

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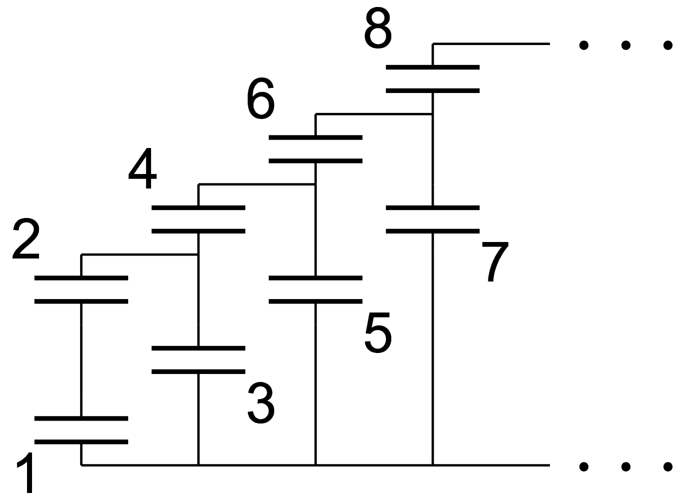
February 24, 2000

1 Assigned Problem



A group of identical $C = 1\,\mu\text{F}$ capacitors are connected recursively, as shown above. Calculate the equivalent capacitance of (a) the first two capacitors, (b) the first three capacitors. (c) Repeat, up to the eighth capacitor. (d) (*Optional.*) In the limit of infinitely many capacitors, what is the equivalent capacitance of this configuration? (*Hint*: $\lim_{n\to\infty} C_{eq}(n) = C_{eq}(n+2)$.)

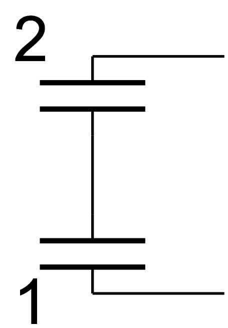
2 Solution

2.1 Part (a)

As our notation, let's define

$$C_{eq}(n) = C_{1+2+\cdots+n}. \tag{1}$$

as the equivalent capacitance of the first n capacitors.

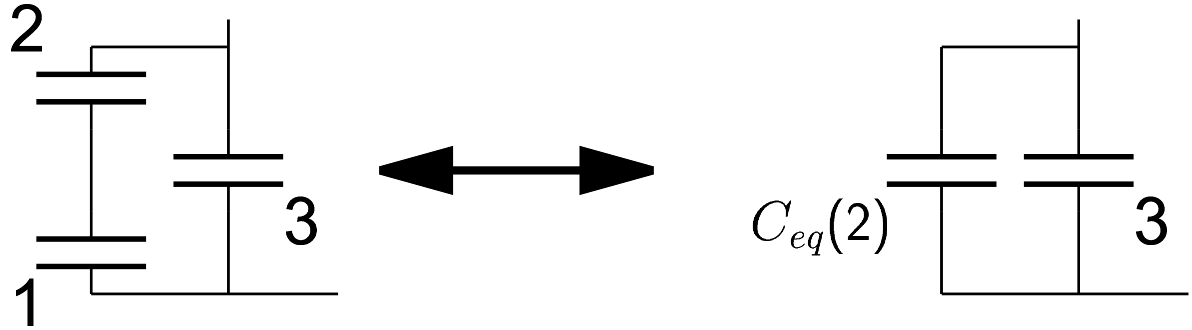


OK, this is pretty easy...just two capacitors in series so

$$C_{eq}(2) \;=\; \left[\frac{1}{C} + \frac{1}{C}\right]^{-1} \tag{2}$$

$$= C/2 \quad (3)$$
$$= 1/2\,\mu\text{F}. \quad (4)$$

2.2 Part (b)



The first two capacitors can be represented as a single capacitor now, so this time we just have to calculate the equivalent capacitance of two capacitors in parallel

$$C_{eq}(3) = C + C_{eq}(2) \quad (5)$$
$$= 1 + 1/2 \quad (6)$$
$$= 3/2\,\mu\text{F}. \quad (7)$$

2.3 Part (c)

If we have already calculated the equivalent capacitance $C_{eq}(n-1)$ of the first $n-1$ capacitors then it is pretty easy to calculate it for the n'th capacitor. If n is *even* then the n'th capacitor will be added in series and if n is *odd* then it will be added in parallel so

$$C_{eq}(n) = \begin{cases} \left[1 + \frac{1}{C_{eq}(n-1)}\right]^{-1} & n \text{ even} \\ 1 + C_{eq}(n-1) & n \text{ odd.} \end{cases} \tag{8}$$

Now we can just cruise through and fill in the numbers...

n	$C_{eq}(n)/(\text{in } \mu\text{F})$
2	1/2
3	3/2
4	3/5
5	8/5
6	8/13
7	21/13
8	21/34

Aside: Notice anything interesting? If you shift between numerator and denominator as you read down the right-hand column the numbers are of the form $F_n = F_{n-1} + F_{n-2}$, the famous Fibonacci sequence! Weird, huh?

2.4 Part (d)

We can see from the above table that the evens and odds each seem to be converging to some constants, C^{∞}_{even} and C^{∞}_{odd} respectively, as the number of capacitors grow.

In the limit as $n \to \infty$ we expect they will each converge so that the solution for n will be identical to $n-2$, that is

$$\lim_{n\to\infty} C_{eq}(n) = C_{eq}(n-2). \tag{9}$$

So let's see what that works out to for n even and n odd.

2.4.1 Even n

Applying Eq. (8) twice, first with even n and again with odd $n-1$ gives

$$C_{eq}(n) = \left[1 + \frac{1}{1 + C_{eq}(n-2)}\right]^{-1} \tag{10}$$

which, when $C_{eq}(n) = C_{eq}(n-2) = C^{\infty}_{\text{even}}$, forms the equation

$$C^{\infty\,2}_{\text{even}} + C^{\infty}_{\text{even}} - 1 = 0 \tag{11}$$

which has a positive solution

$$C^{\infty}_{\text{even}} = \frac{\sqrt{5}-1}{2} \approx 0.618\,\mu\text{F}. \tag{12}$$

2.4.2 Odd n

Conversely, applying Eq. (8) twice, first with odd n and again with even $n-1$ gives

$$C_{eq}(n) = 1 + \left[1 + \frac{1}{C_{eq}(n-2)}\right]^{-1} \tag{13}$$

which, when $C_{eq}(n) = C_{eq}(n-2) = C^{\infty}_{\rm odd}$, forms the equation

$$C^{\infty\,2}_{\rm odd} - C^{\infty}_{\rm odd} - 1 = 0 \tag{14}$$

which has a positive solution

$$C^{\infty}_{\rm odd} = \frac{\sqrt{5}+1}{2} \approx 1.618\,\mu\text{F}. \tag{15}$$

So there are two solutions to the problem of the equivalent capacitance as the number of capacitors grow, depending on whether we stop on an even or odd number.

Aside: You probably noticed that $C^{\infty}_{\rm odd} = C^{\infty}_{\rm even} + 1$ but did you notice that $C^{\infty}_{\rm odd} = 1/C^{\infty}_{\rm even}$, too? Blows your mind, eh? (Well…it's cool to physics geeks like me!) These are called the golden section numbers. They are often observed in nature. For more interesting tidbits on Fibonacci numbers and the golden section visit `http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`.