Physics 153 Section T0H - Solution to Tipler Ch. 25 #60.

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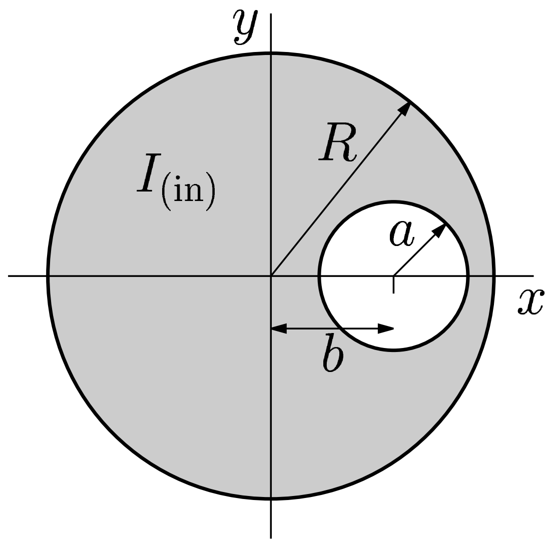
1 Problem



A very long, straight conductor with a circular cross section of radius R carries a current I. Inside the conductor, there is a cylindrical hole of radius a whose axis is parallel to the axis of the conductor a distance b from it (figure above). Let the z axis be the axis of the conductor, and let the axis of the hole be at $x = b$.

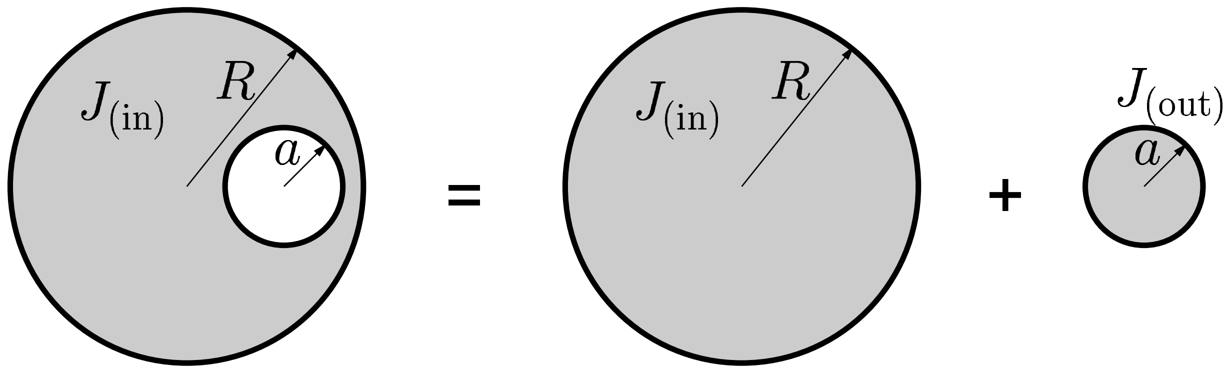
Show that the magnetic field inside the hole is uniform, and find its magnitude and direction.

2 Solution

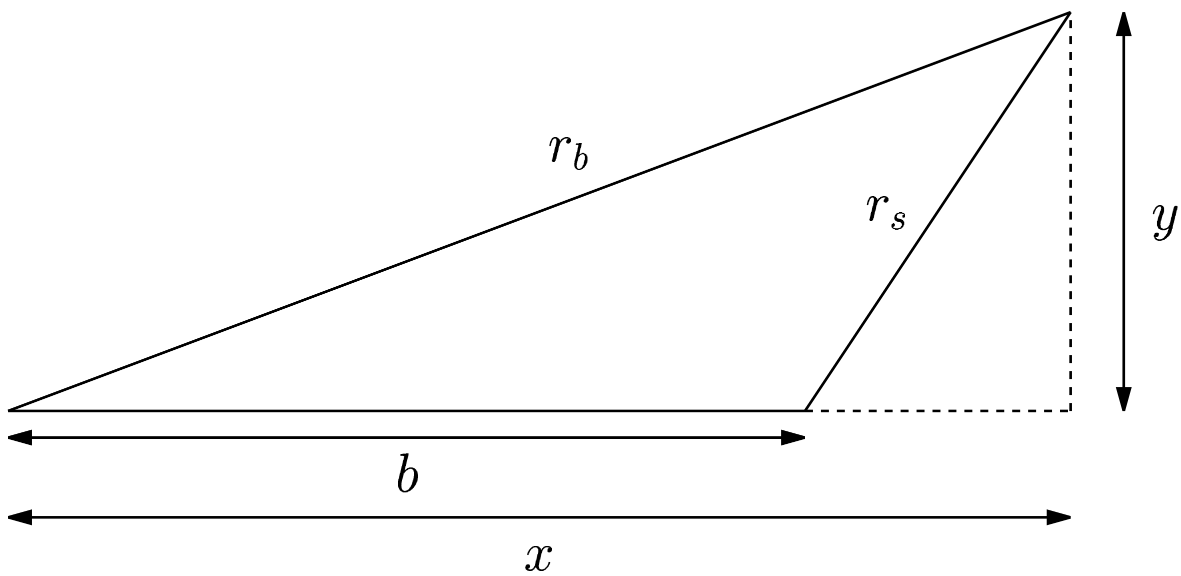
First, let's calculate the current density J in the wire, given by the total current divided by the total cross-sectional area:

$$J = \frac{I}{A} = \frac{I}{\pi(R^2 - a^2)}. \tag{1}$$

Now the trick to solving the problem is to treat the wire with the hole as the superposition of two wires, a big one with current density J going into the page and a small one with current density J coming out of the page (representing the hole). Then, when we superpose these the current densities cancel out in the hole ($J + -J = 0$) and we get the original wire.



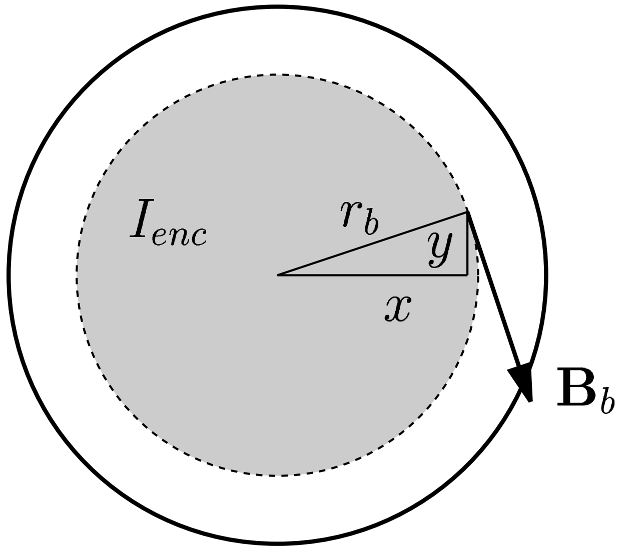
Now we consider an arbitrary point (x, y) (measured from the center of the big wire) inside the hole. The distance from the axes of the big and small wires are r_b and r_s, respectively.



Now we use Ampère's law to calculate the magnetic field due to the big wire and the small wire.

2.1 Big wire

The natural Ampèrian loop is a coaxial circle:



The loop encloses a current (into the page)

$$I_{enc} = \pi r_b^2 J \tag{2}$$

so Ampère's law reduces to

$$2\pi r_b B_b = \mu_0 I_{enc} = \pi r_b^2 J \tag{3}$$

which gives a magnetic field strength due to the big wire

$$B_b = \frac{1}{2}\mu_0 J r_b. \tag{4}$$

But we need to find the magnetic field *vector* so we have to figure out the direction $\mathbf{B}_b$ is pointing. I'll just write down the answer and let you puzzle over it for yourself but the basic idea is that r_b and $\mathbf{B}_b$ form similar triangles except for a rotation. So the unit vector $\hat{\mathbf{B}}_b$ is

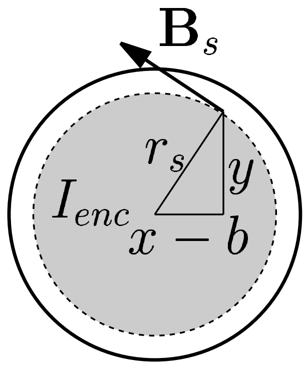
$$\hat{\mathbf{B}}_b = \frac{y}{r_b}\mathbf{i} - \frac{x}{r_b}\mathbf{j}. \tag{5}$$

So the magnetic field due to the big wire is

$$\begin{aligned} \mathbf{B}_b &= \frac{1}{2}\mu_0 J r_b \left[\frac{y}{r_b}\mathbf{i} - \frac{x}{r_b}\mathbf{j}\right] \qquad (6) \\ &= \frac{1}{2}\mu_0 J \left[y\mathbf{i} - x\mathbf{j}\right]. \qquad (7) \end{aligned}$$

2.2 Small wire

Again, we use a circle for our Ampèrian loop:



The loop encloses a current (out of the page)

$$I_{enc} = \pi r_s^2 J \tag{8}$$

so Ampère's law reduces to

$$2\pi r_s B_s = \mu_0 I_{enc} = \pi r_s^2 J \tag{9}$$

which gives a magnetic field strength due to the small wire

$$B_s = \frac{1}{2}\mu_0 J r_s. \tag{10}$$

We calculate the direction the same way we did before except the current is out of the page so (from the Right-hand rule) the magnetic field is counter-clockwise:

$$\hat{\mathbf{B}}_s = -\frac{y}{r_s}\mathbf{i} + \frac{x-b}{r_s}\mathbf{j}. \tag{11}$$

So the magnetic field due to the small wire is

$$\mathbf{B}_s = \frac{1}{2}\mu_0 J r_s \left[-\frac{y}{r_s}\mathbf{i} + \frac{x-b}{r_s}\mathbf{j} \right] \tag{12}$$

$$= \frac{1}{2}\mu_0 J \left[-y\mathbf{i} + (x-b)\mathbf{j} \right]. \tag{13}$$

2.3 Net magnetic field

The total magnetic field at (x, y) is given by the (vector) sum of the two contributions

$$\mathbf{B} = \mathbf{B}_b + \mathbf{B}_s \tag{14}$$

$$= \frac{1}{2}\mu_0 J \left[y\mathbf{i} - x\mathbf{j} - y\mathbf{i} + (x-b)\mathbf{j} \right] \tag{15}$$

$$= -\frac{1}{2}\mu_0 J b\,\mathbf{j} \tag{16}$$

$$= -\frac{\mu_0 I b}{2\pi(R^2 - a^2)}\mathbf{j} \tag{17}$$

which is uniform over the entire hole! Surprising, huh? It points straight down and has the same magnitude everywhere in the hole.