UBC ISCI 344 Game Theory
Symmetric and zero-sum games
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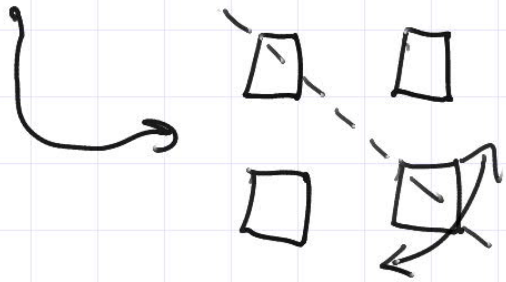
Outline:
- symmetric games
- zero-sum (or constant-sum) games

Symmetric games:

- Row and column players play same game

→ Payoff matrix same after swapping roles

$$\text{Rikky}\ \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Chrissy}} \\ & L & R \\ \hline U & a,\alpha & b,\gamma \\ D & c,\beta & d,\delta \end{array} \quad \xrightarrow[\text{roles}]{\text{swap}} \quad \text{Chrissy}\ \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Rikky}} \\ & U & D \\ \hline L & \alpha, a & \beta, c \\ R & \gamma, b & \delta, d \end{array}$$



transpose, or flip across diagonal

- reflection symmetry: matrix unchanged after flip

→ Strategies unchanged: $L=U$, $R=D$

Payoffs unchanged: $a=\alpha$, $b=\beta$, $c=\gamma$, $d=\delta$

- symmetric games must be square, ie. for $M\times N$ game, $M=N$

- simplified payoff matrix

$$\begin{array}{c|cc} & U & D \\ \hline U & a & b \\ D & c & d \end{array} \xrightarrow{\text{symmetric}} \begin{array}{c|cc} & U & D \\ \hline U & a, a & b, c \\ D & c, b & d, d \end{array}$$

Zero-sum (or constant-sum) games:

- Sum of all players' payoffs same for all outcomes

→ a higher payoff for one player comes at a cost to other player(s)

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{Chrissy}} \\ & & L & R \\ \hline \text{Rikky} & U & a, \alpha & b, \beta \\ & D & c, \gamma & d, \delta \end{array}$$

Zero-sum:

$a + \alpha = 0$

$b + \beta = 0$

$c + \gamma = 0$

$d + \delta = 0$

- simplified payoff matrix:

$$\begin{array}{c|cc} & L & R \\ \hline U & a & b \\ D & c & d \end{array} \xrightarrow{\text{zero-sum}} \begin{array}{c|cc} & L & R \\ \hline U & a, -a & b, -b \\ D & c, -c & d, -d \end{array}$$

Constant-sum games:

- $a + \alpha = b + \beta = c + \gamma = d + \delta = K$ (constant)

- rational choice only depends on <u>differences</u> between payoffs
  - → can subtract off constant from every outcome without changing game
  - → every constant-sum game is zero-sum game

Ex.

	L	R
U	20, 80	20, 80
D	50, 50	80, 20

constant-sum:

$a+\alpha = 20+80 = 100$

$b+\beta = c+\gamma = d+\delta = 100$

- subtract 100 from row player:

  $a' = a-100,\ b' = b-100,\ c' = c-100,\ d' = d-100$

	L	R
U	-80, 80	-80, 80
D	-50, 50	-20, 20

zero-sum:

$a'+\alpha = b'+\beta = c'+\gamma = d'+\delta = 0$

→ could've subtracted 100 from column player or any x from row and y from column as $x+y=100$

- solution concepts always predict same outcomes for constant-sum and equivalent zero-sum games

> Convention: Unless otherwise stated, simplified matrices will indicate <u>symmetric</u> games.

Summary:
- symmetric games
- simplified payoff matrix
- zero-sum games
- constant-sum games
- convention: simplified $\Rightarrow$ symmetric by default