UBC ISCI 344 Game Theory
Deriving mixed Nash equilibria

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Outline:
- general 2×2 payoff matrix
- maximize expected utility
  1) graphically
  2) standard optimization
  3) endpoints shortcut
- interpretation

General 2×2 payoff matrix:

- recall Nash's existance theorem: every finite game has at least one Nash equilibrium (NE)
- mixed strategies: Rikky plays I with prob. p, else II.
  Chrissy " 1 " q " 2.

Payoffs:

		Chrissy 1	Chrissy 2
Rikky	I	a, α	b, β
Rikky	II	c, γ	d, δ

Probabilities:

		Chrissy q	Chrissy $1-q$
Rikky	p	pq	$p(1-q)$
Rikky	$1-p$	$(1-p)q$	$(1-p)(1-q)$

- expected utilities:
  - Rikky: $U_R = pqa + p(1-q)b + (1-p)qc + (1-p)(1-q)d$
  - Chrissy: $U_C = pq\alpha + p(1-q)\beta + (1-p)q\gamma + (1-p)(1-q)\delta$

Maximize Rikky's expected utility:

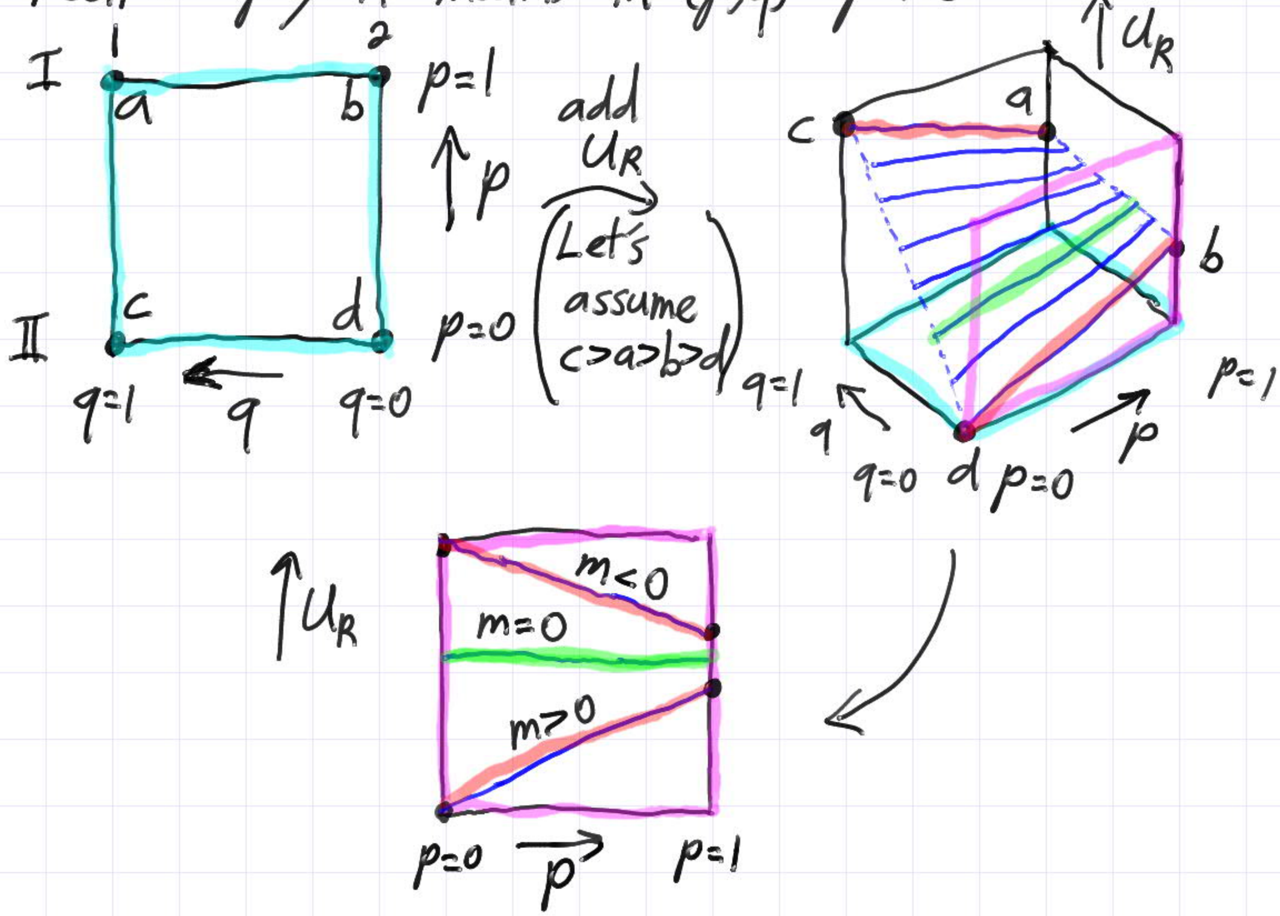
1) Graphically:

- notice U_R linear function of p:

$$\underset{\uparrow}{\overset{}{U_R}} = \underbrace{\left[qa + (1-q)b - qc - (1-q)d\right]}_{m} \underset{\uparrow}{p} + \underbrace{qc + (1-q)d}_{b}$$

$$y = \qquad m \qquad x + \qquad b$$

- but m and b depend q: $m = m(q)$, $b = b(q)$
- redraw payoff matrix in (p,q)-space



- generally, there is a special q^* where $m(q^*) = 0$
  (exception: if $m(q)$ = constant, doesn't depend on q.)

$$m(q^*) = 0 = q^*a + (1-q^*)b - q^*c - (1-q^*)d$$

$$\longrightarrow q^* = \frac{b-d}{c-a+b-d}$$

- when $q = q^*$ Rikky has no preference for p (indifferent)
- when $q \neq q^*$ Rikky always has a clear preference, $p=0$ or $p=1$
- Exercise: do same derivation for Chrissy.

  Q: What value of p^* makes Chrissy indifferent?

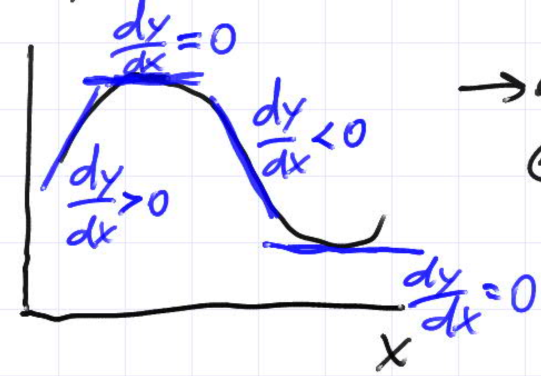
  A:

$$p^* = \frac{\gamma - \delta}{\beta - \alpha + \gamma - \delta}$$

Maximize Rikky's expected utility:

2) Standard optimization

[Aside:



→ extrema (max/min) @ $\frac{dy}{dx} = 0$]

- Rikky can optimize U_R by requiring zero slope as a function of p:

$$\frac{dU_R}{dp} = qa + (1-q)b - qc - (1-q)d$$

$$= 0 \longrightarrow q^* = \frac{b-d}{c-a+b-d}$$

- is it a maximum or minimum? A: Both, because U_R = constant when $q = q^*$. Doesn't depend on p.
- when $q = q^*$ Rikky is indifferent — any p gives same payoff

- Exercise: repeat for Chrissy.

  Q: what is condition that optimizes Chrissy's utility:

  A: Chrissy becomes indifferent when

$$p^* = \frac{\gamma - \delta}{\beta - \alpha + \gamma - \delta}$$

Maximize Rikky's expected utility:

3) Endpoints shortcut

- Chrissy chooses q^* to make Rikky indifferent to pure strategies, $p=0$ or $p=1$

$$U_R(0,q^*) = U_R(1,q^*)$$

$$0 + 0 + q^*c + (1-q^*)d = q^*a + (1-q^*)b + 0 + 0$$

$$\longrightarrow q^* = \frac{b-d}{c-a+b-d} \qquad \text{Same!}$$

- Exercise: Do same derivation to find p^*.

  $\longrightarrow$ works because if U_R the same at endpoints ($p=0,1$) then same for <u>all</u> p.

Interpretation:

- Chrissy can choose q^* to make Rikky indifferent and Rikky " p^* " Chrissy " .

  $\longrightarrow$ Rikky should choose $q=q^*$?

  No! q is Chrissy's strategy!

→ (p^*, q^*) is a mixed NE because both are indifferent (no incentive to switch)

Summary:
- general 2×2 payoff matrix
- mixed strategies
- how to maximize expected utility
  1) graphical
  2) standard optimization
  3) endpoints shortcut
- mixed NE
- twist: playing mixed NE strategy makes <u>other</u> player indifferent