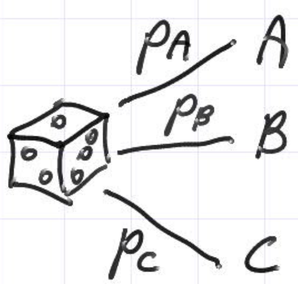
UBC ISCI 344 Game Theory
Deriving Expected Utility Theory
Rik Blok and Christoph Hauert

Outline:
- lotteries
- derive utility
- derive expected utility theory
- implications
- rescaling utilities

Lotteries:

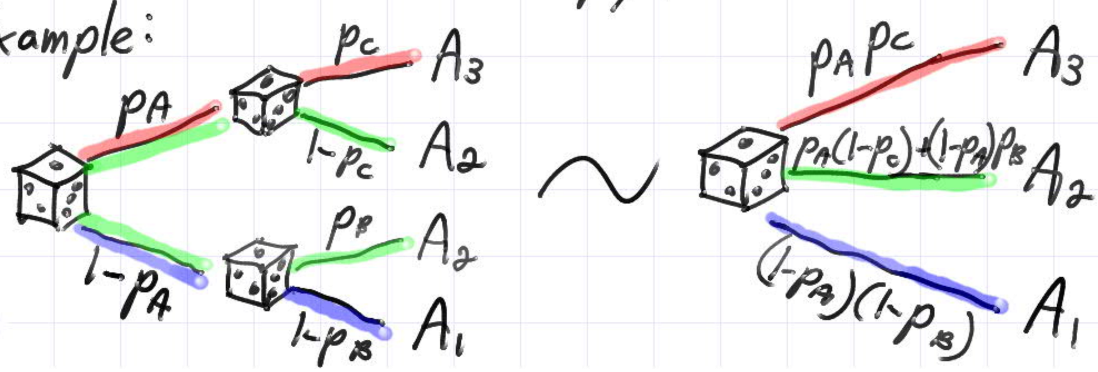
- expected utility theory about how to make decisions when outcomes uncertain → lotteries convenient framework
- multiple possible outcomes determined by chance



p_A, p_B, p_C probabilities
so $0 \leq p_A, p_B, p_C \leq 1$
and $p_A + p_B + p_C = 1$

- branches can be nested. Normal "And/Or" rules of probability apply
- branches are independent and exclusive
  → "And" means multiply, "Or" means add

Example:

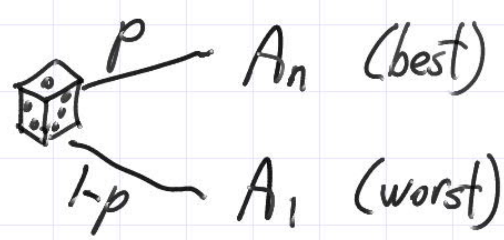


Derive utility:

- consider n outcomes $A_1, \ldots, A_n$ where A_n is most preferred and A_1 is least
- going to show that outcomes can be assigned numbers and preferences ranked by numbers
  - → utility: a measure of preference

- create "extreme" lottery for the best vs. worst:
  $X(p)$ = lottery that gives A_n (best) with probability p otherwise gives A_1 (worst).
  - → $X(0)$ same as A_1 and $X(1)$ same as A_n

$X(p)$:



- for every outcome A_i there is a lottery $X(u_i)$ with probability u_i that makes player indifferent: $A_i \sim X(u_i)$

→ special cases: $A_1 \sim X(0) \longrightarrow u_1 = 0$
$\qquad\qquad\qquad\;\; A_n \sim X(1) \longrightarrow u_n = 1$

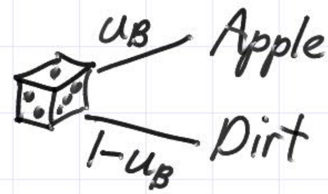
- $X(u_i)$ preferred to $X(u_j)$ if and only if $u_i > u_j$
- then outcome A_i preferred to A_j if and only if $u_i > u_j$

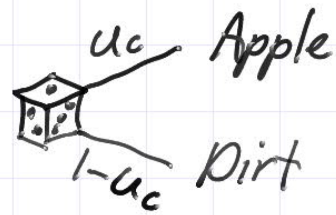
→ utility: every outcome A_i can be assigned value u_i which is probability of lottery with same preference

→ larger utility means stronger preference

- everybody can assign different utilities — this result just shows there <u>are</u> numeric utilities

Example: If you prefer an Apple (A) over a Banana (B) over a Carrot (C) over Dirt (D) then

- Banana $\sim$ 

- Carrot $\sim$ 

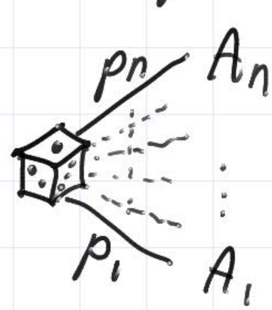
- $u_A = 1$, $u_D = 0$
- $0 < u_B < 1$, $0 < u_C < 1$
- $u_B > u_C$ because Banana preferred to Carrot

Derive expected utility theory:

- how to choose between uncertain outcomes
- let's create new lottery for all outcomes:

  $L(p_1, p_2, \ldots, p_n)$ = lottery that gives A_i with probability p_i

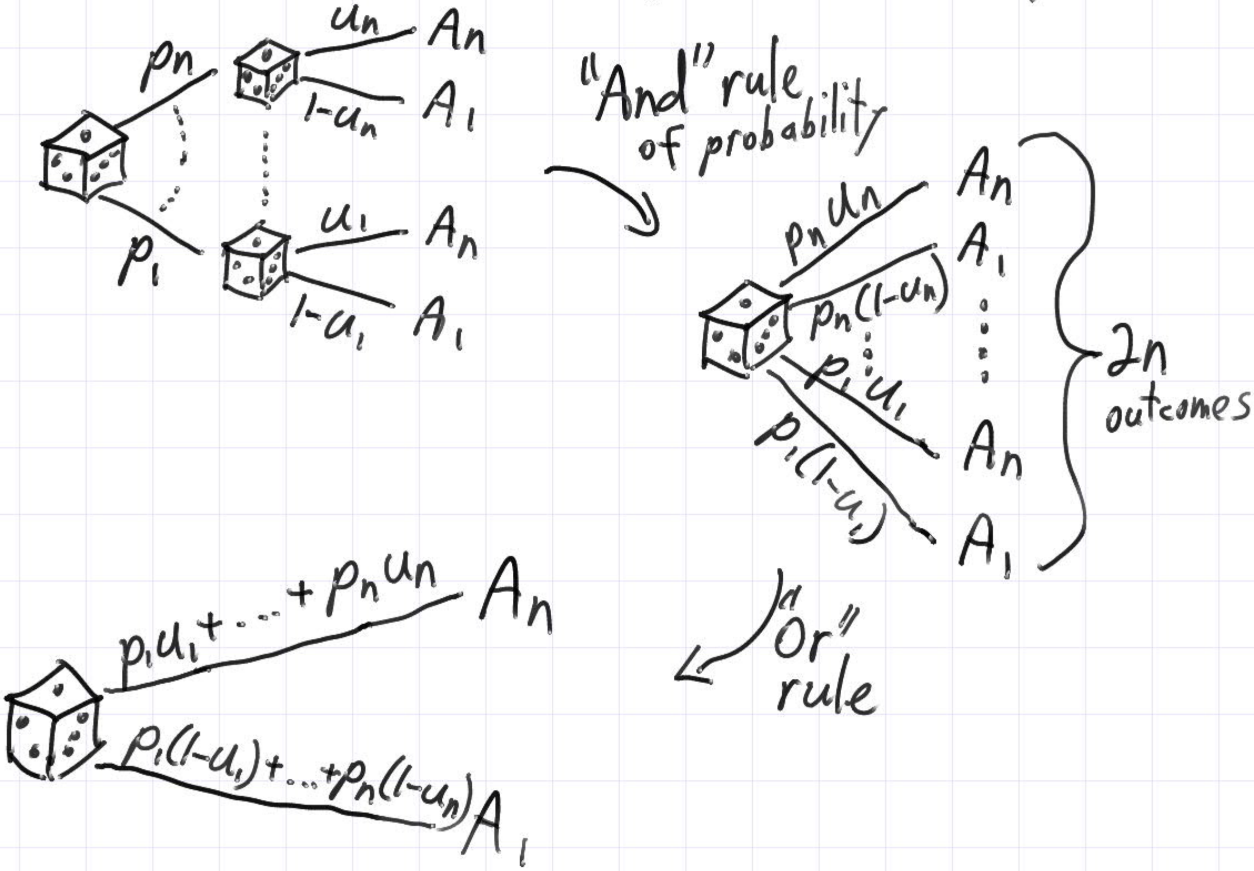
  → must have $p_1 + \cdots + p_n = 1$

$L(p_1, \ldots, p_n)$: 

- how to choose between two lotteries:

  $L(p_1, \ldots, p_n)$ vs. $L(q_1, \ldots, q_n)$

- recall every outcome A_i already paired with lottery $X(u_i)$



- so any lottery L same as extreme lottery X with specially chosen probability: $L(p_1,\ldots p_n) \sim X(p_1u_1+\ldots+p_nu_n)$
- already saw higher probability preferred in extreme lottery: $X(p)$ preferred over $X(q)$ if and only if $p>q$

$\rightarrow$ $L(p_1,\ldots p_n)$ preferred over $L(q_1\ldots q_n)$ if and only if

$$p_1u_1+\ldots+p_nu_n > q_1u_1+\ldots+q_nu_n$$

$\rightarrow$ expected utility: sum of utilities weighted by probabilities, $EU[L(p_1,\ldots p_n)] = p_1u_1+\ldots+p_nu_n$

$\rightarrow$ any uncertain outcome (lottery) can be assigned expected utility and choosing highest possible expected utility reflects preferences

Implications:

- derivation is informal proof. If assumptions and steps accepted then conclusion is inescapable
  → this is how rational individuals <u>must</u> behave
- only shows how to choose once utilities of outcomes (u_i) assigned — doesn't say how to choose u_i
  (come from personal preferences)

Rescaling utility:

- expected utility tells us to maximize $EU = p_1 u_1 + \cdots + p_n u_n$
- in derivation we assigned $u_1 = 0$ and $u_n = 1$
- but can rescale: choose rescaling factor $m > 0$ and shift by b

$$
\begin{aligned}
EU' &= m(EU) + b \\
&= m(p_1 u_1 + \ldots + p_n u_n) + b \\
&= m\left(\sum p_i u_i\right) + b \\
&= m\left(\sum p_i u_i\right) + b \sum p_i \\
&= \sum (m p_i u_i + b p_i) \\
&= \sum p_i (m u_i + b) \\
&= p_1 u_1' + \ldots + p_n u_n' \quad \text{where } u_i' = m u_i + b
\end{aligned}
$$

→ can multiply utilities by any positive factor m and add any b without changing preferences

→ just have to rescale <u>all</u> of a player's utilities in the same way

Summary:
- lotteries
- derive utility for certain outcomes
- derive expected utility theory for uncertain outcomes
- implications
- rescaling utility