# ISCI 344 Game Theory
# Public goods with punishment

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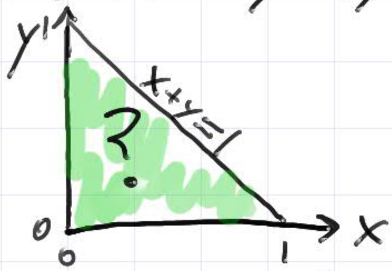
Outline:
- Public goods game
- problem of cooperation
- peer punishment
- pairwise invasion: D vs. C, C vs. P, P vs. D
- pairwise invasion graph
- evolutionary stable strategy (ESS)
- ESS in economic game theory

## Public goods game:
- group of $N$ players, either C=cooperator or D=defector
- C's contribute cost $c$ to public good
- public good grows by factor $r$, $1<r<N$
- distributed to <u>all</u> group members

## Problem of cooperation:
- let $x$ = frequency of C in population
- fitness: $f_D = x(N-1)cr/N$
  $f_C = (x(N-1)+1)cr/N - c < f_D$
- how can cooperation evolve?

## Peer punishment:
- add third strategy: peer punishers, P
  - → cooperate like C
  - → also impose fine $\beta$ (beta) on each D, at a cost $\gamma$ (gamma) per D.

- let $y$ = frequency of $P$ in population
  → $1-x-y$ = frequency of $D$ $(x+y \le 1)$
- general solution for any $x,y$ — too difficult

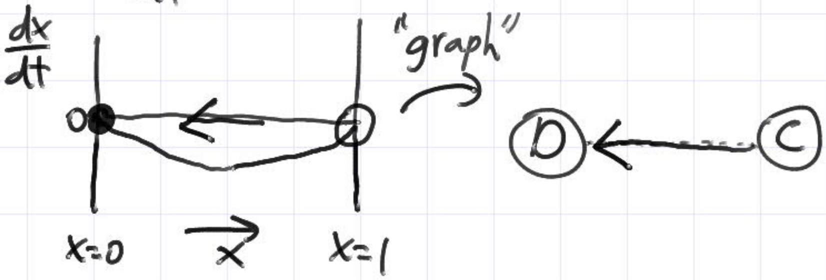


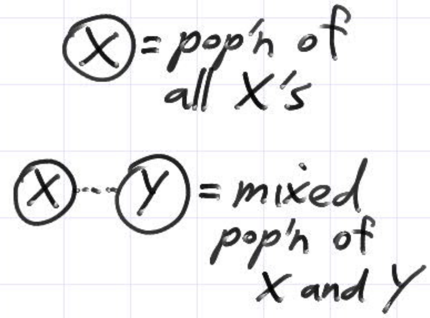

- instead, just consider mixed populations of 2 types

Pairwise invasion, D vs. C:

- assume just D's and C's, no P's → $y=0$

$$\frac{dx}{dt} = x(1-x)(f_C - f_D) \quad \text{where } f_C < f_D$$



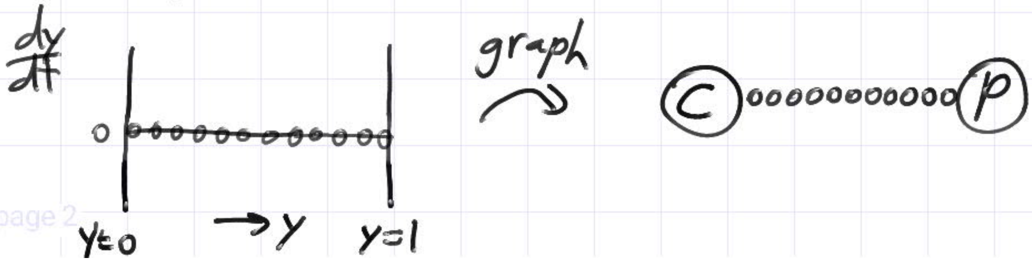

Pairwise invasion, C vs. P:

- assume just C's and P's, no D's → $x+y=1$

$$\frac{dy}{dt} = y(1-y)(f_P - f_C)$$

$$f_C = \frac{Nrc}{N} - c \,, \quad f_P = \frac{Nrc}{N} - c - \cancel{y(0)}^{\to 0} = f_C$$

$$\frac{dy}{dt} = y(1-y)(0) = 0 \quad \text{neutral!}$$



# Pairwise invasion, P vs. D:

- pop'n of just P's and D's, no C's $\longrightarrow x=0$

$$\frac{dy}{dt} = y(1-y)(f_P - f_D)$$

$$f_D = \frac{y(N-1)rc}{N} - \beta y(N-1)$$ ← defectors fined by punishers

$$f_P = \frac{(y(N-1)+1)rc}{N} - c - \gamma(1-y)(N-1)$$ ← punishers paying cost to fine defectors

$$f_P - f_D = \underbrace{-\left(1-\frac{r}{N}\right)c - \gamma(N-1)}_{\ominus} + \underbrace{(\beta+\gamma)(N-1)}_{\oplus} y$$

- $f_P = f_D$ when

$$y^* = \frac{\left(1-\frac{r}{N}\right)c + \gamma(N-1)}{(\beta+\gamma)(N-1)}$$

- $y^* > 0$. Is $y^* < 1$?

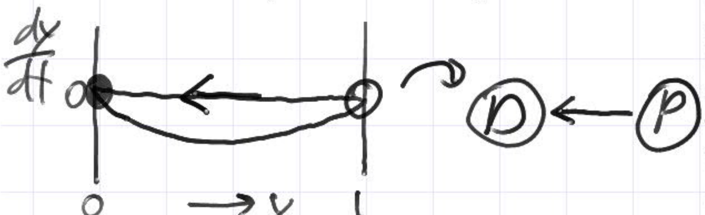
"defined as"

$$y^* < 1 \longrightarrow \beta > \beta_{min} \equiv \frac{\left(1-\frac{r}{N}\right)c}{N-1}$$

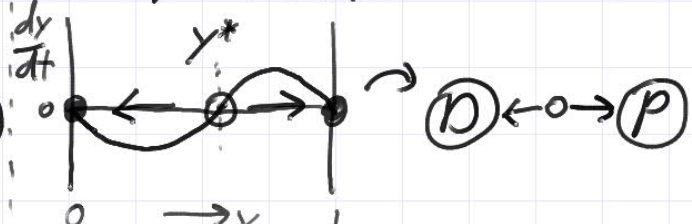
→ interior fixed point, $0 < y^* < 1$, if $\beta$ (fine) large enough

- 2 cases:

$\beta < \beta_{min}$



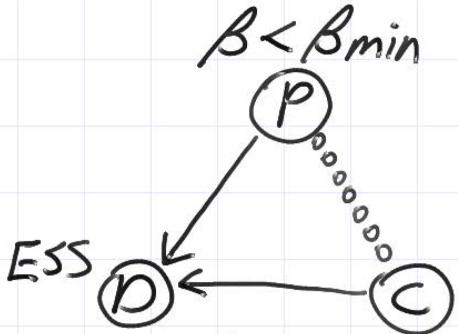

$\beta > \beta_{min}$

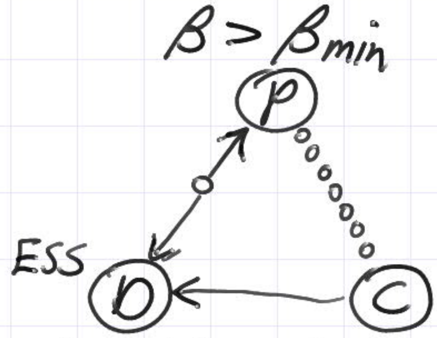



# Pairwise invasion graph (PIG):

- draw all pure populations as nodes: Ⓓ, Ⓒ, Ⓟ
- draw arrows between each pair to show evolution

- 2 cases:



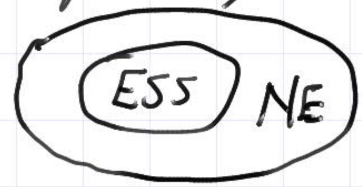

- can understand evolution (partly) just by looking at directions of links between nodes
  - → shows which strategies can/can't be invaded

Evolutionary stable strategy (ESS):

- a strategy that is a stable equilibrium when adopted by whole population
  - → can't be invaded by any other strategy
  - → all pairwise invasion arrows point inwards
- Ⓓ is an ESS for both $\beta < \beta_{min}$ and $\beta > \beta_{min}$
- is Ⓟ also an ESS when $\beta > \beta_{min}$?
  - → no, neutral along Ⓟ∘∘∘∘Ⓒ edge
  - → population may stay near Ⓟ for long time but will eventually drift to Ⓒ. Then evolves Ⓟ and stays.
- conclusion: peer punishment can prolong cooperation if fine is big, but eventually D's will take over
  - → peer punishment can't protect a public good
  - → "second-order" free riding cooperators undermines punishment

# Aside: ESS and PIGs in economic game theory

- ESS and PIGs from evolutionary game theory (GT)
- also relevant to economic GT (symmetric games)
- fitness = payoff
- (X) = "if everybody *else* plays X..."
- arrows show preferences
  - (X) ← = if everybody else plays X, I should play X.
  - (X) → (Y) = " , I should play Y.
- holds for all players, so shows reasoning
- ESS are nodes where nobody can improve by unilaterally switching
  - → every ESS is a NE
  - → but not every NE is an ESS




**Summary:**
- public goods game
- problem of cooperation
- peer punishers → pay cost $\gamma$ to fine D's $\beta$
- pairwise invasion: D vs. C, C vs. P, P vs. D
- pairwise invasion graph (PIG)
- evolutionary stable strategies (ESS)
  - → found (D) still only ESS, peer punishment can't maintain cooperation
- ESS (and PIGs) in economic game theory