On the Expected ℓ_∞-norm of High-dimensional Martingales

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Abstract

Motivated by a problem from theoretical machine learning, we show asymptotically optimal bounds on $\mathrm{E}[\,\|X_\tau\|_\infty\,] / \mathrm{E}[\,\sqrt{\tau}\,]$, where $(X_t)_{t\geq 0}$ is a continuous stochastic process in $\mathbb{R}^n$ with $(X_{t,i})_{t\geq 0}$ being a Brownian motion for each $i \in \{1,\ldots,n\}$ and τ being a stopping time such that $\mathrm{E}[\,\sqrt{\tau}\,] < \infty$. We further extend this result to the setting where the entries of $(X_t)_{t\geq 0}$ have smooth quadratic variation. Finally, we show a similar result for discrete-time processes using analogous techniques, together with a discrete version of Itô's formula.

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1. Introduction

Let $(X_t)_{t\geq 0}$ be a stochastic process in $\mathbb{R}^n$ such that $(X_{t,i})_{t\geq 0}$ is a standard Brownian motion for each $i \in [n] \coloneqq \{1,\ldots,n\}$. In this paper, we study the values of

$$K_n \coloneqq \sup_\tau \frac{\mathrm{E}[\,\|X_\tau\|_\infty\,]}{\mathrm{E}[\,\sqrt{\tau}\,]} \tag{1}$$

for all values of n, where the above supremum is taken over all stopping times τ (defined with respect to the same filtration used for the process $(X_t)_{t\geq 0}$) such that $\mathrm{E}[\,\sqrt{\tau}\,] < \infty$.

Our initial motivation for investigating K_n comes from the study of a continuous-time model of *online learning* [7, 4], a theoretical model of machine learning over a stream of data examples. In this model, at each instant in time $t \geq 0$ a learner picks weights $p(t) \in [0,1]^n$ that sum to 1 over a set of n actions or "experts". Moreover, the experts are given rewards at each time instant t that reflect their performance at that instant. The instantaneous reward of the learner is given by the experts' rewards weighted by $p(t)$. In the continuous-time setting, the vector of rewards of the experts is given by a continuous stochastic process

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$(X_t)_{t \geq 0}$ in $\mathbb{R}^n$ where the reward process $(X_{i,t})_{t \geq 0}$ of each expert $i \in [n]$ follows a Brownian motion, although the rewards of different experts need not be independent. Past work on this setting focused on finding almost sure bounds on the performance of strategies for the learner. Interestingly, the *expected* performance of any algorithm at a stopping time τ with $\mathrm{E}[\sqrt{\tau}] < \infty$ is $\mathrm{E}[\|X_\tau\|_\infty]$ by the optional stopping theorem (see [7, §5.2] for a similar discussion), which inspired the study of K_n.

Problems related to (1) were studied in the past, but mostly focusing on the expected ℓ_2-norm of X_t. For example, Burkholder [1] and Davis [2, 3] studied the convergence of $\|X_\tau\|_2$ to $\sqrt{\tau n}$ for any stopping time τ with $\mathrm{E}[\sqrt{\tau}] < \infty$ as $n \to \infty$. Of particular interest for us is the work of Davis [2] on the case with $n = 1$. In this work, he studies *optimal* bounds on $\mathrm{E}[|X_\tau|^p] / \mathrm{E}\left[\tau^{p/2}\right]$ for stopping times τ such that $\mathrm{E}\left[\tau^{p/2}\right] < \infty$ and $0 < p < \infty$. One of Davis' results is that, for $n = 1$, $\mathrm{E}[|X_\tau|] \leq \lambda(0)\,\mathrm{E}[\sqrt{\tau}]$ for any stopping time τ, where $\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the inverse of a function that involves a confluent hypergeometric function, and that this constant cannot be improved. In this note we give asymptotically tight bounds on K_n with techniques similar to the ones used by Davis in [2]. Namely, we show

$$K_n \leq \lambda(n-1) \leq 3 + \sqrt{2 \ln n}, \qquad \forall n \geq 1, \tag{2}$$

where the bounds on $\lambda(n)$ and its asymptotic behavior as $n \to \infty$ may be of independent interest since the confluent hypergeometric function involved and its inverse appear in previous work on related problems [2, 7, 5, 11]. The bound in (2) can be seen as a generalization of Davis' results to higher dimensions. Furthermore, we know that for any $\varepsilon > 0$ and for sufficiently large n, we have $\mathrm{E}[\|X_t\|_\infty] \geq (1-\varepsilon)\sqrt{2t \ln n}$ for any $t \geq 1$ due to the asymptotics of the expected value of the maximum of n Gaussian random variables (e.g., see [13, Exercise 2.11] or [10, Theorem 3]). The asymptotic behavior of λ is $\lim_{n \to \infty} \frac{\lambda(n-1)}{\sqrt{2 \ln n}} = 1$, and so (2) is asymptotically tight as $n \to \infty$. We provide a slightly improved lower bound: we show that there are constants c_n for $n \geq 1$ with $c_n / \sqrt{2 \ln n} \to 1$ as $n \to \infty$ such that $c_n \leq K_n$ for all $n \geq 1$ and, for large n, $c_n > \sqrt{2 \ln n}$ and $\lambda(n-1) - c_n \leq 3$. Interestingly, this result follows from a careful choice of a stopping time for a standard n-dimensional Browninan motion. Showing more quantitative lower bounds on c_n and pinning down the exact values of K_n are interesting open questions. Finally, we extend the upper bounds on K_n to discrete-time martingales using remarkably similar techniques thanks to a discrete version of Itô's formula and some properties of the confluent hypergeometric function. While our results hold for any martingale with bounded increments, Davis [2] has results for discrete-time martingales of a different class: those whose increments have distributions symmetric around 0. We also pose a conjecture that yields better bounds for martingales with increments supported on intervals that are not symmetric around zero.

In Section 2 we prove the upper bounds on K_n first for the case when the coordinates of $(X_t)_{t \geq 0}$ are each distributed as a Brownian motion, and we loosen this assumption on Section 2.1. In Section 3 we extend these results to discrete-time martingales. In Section 5 we collect some well-known properties of the confluent hypergeometric function we use, prove bounds and asymptotic behavior of λ, and discuss a conjecture that improves the results for discrete martingales. Finally, in Section 4 we show a lower bound to K_n that is bigger than $\sqrt{2 \ln n}$ for large n.

2. Continuous-time Martingales

In this section we begin by proving (2). Our proof technique is similar to the one used by Davis. More precisely, for the remainder of this paper, we shall use the function f given by

$$f(t,x) \coloneqq \|x\|_\infty - \beta\sqrt{t}, \qquad \forall x \in \mathbb{R}^n, \forall t \geq 0, \tag{3}$$

where $\beta > 0$ is some fixed constant that may depend on n. Moreover, let $(X_t)_{t\geq 0}$ be a continuous-time martingale. For this section, we assume for each $i \in [n]$ that $(X_{i,t})_{t\geq 0}$ is a standard Brownian motion. We shall loosen this assumption in Section 2.1.

If $(f(t,X_t))_{t\geq 0}$ were a (local) supermartingale, then the inequality in (2) would easily follow since we would have $\mathrm{E}[\, f(\tau, X_\tau)\,] \leq \mathrm{E}[\, f(0,0)\,] = 0$ for any bounded stopping time τ by the optional stopping theorem, and the claim for general stopping times would follow from Fatou's lemma. However, f is non-smooth, so it is not clear whether $(f(t,X_t))_{t\geq 0}$ is a supermartingale. Instead, we shall design a potential function Φ such that $\Phi(t,x) \geq f(t,x)$ and $(\Phi(t,X_t))_{t\geq\delta}$ is a local martingale for any fixed $\delta > 0$ and $\lim_{\delta\downarrow 0}\Phi(\delta,0) = 0$.

The potential function we shall use is based on a confluent hypergeometric function M_0 given by

$$M_0(x) \coloneqq e^x - \sqrt{\pi x}\,\mathrm{erfi}(\sqrt{x}), \qquad \forall x \in \mathbb{R},$$

where erfi is the imaginary error function and is given by

$$\mathrm{erfi}(x) \coloneqq \frac{2}{\sqrt{\pi}} \int_0^x e^{z^2}\,\mathrm{d}z, \quad \forall x \in \mathbb{R}.$$

Our potential function Φ will be a separable function, defined below in (5), depending on ϕ given by

$$\phi(t,x) \coloneqq -\sqrt{t} M_0\left(\frac{x^2}{2t}\right), \qquad \forall t > 0, \forall x \in \mathbb{R}.$$

One interesting feature of ϕ is that it satisfies the backwards heat equation (BHE), i.e.,

$$\partial_t\phi(t,x) + \frac{1}{2}\partial_{xx}\phi(t,x) = 0, \qquad \forall t > 0, \forall x \in \mathbb{R}. \tag{4}$$

For details on the above equation and other properties of ϕ, see [6] or Section 5.

It is already known that $(\phi(t,X_{t,i}))_{t\geq\delta}$ is a local martingale for any $i \in [n]$ and fixed $\delta > 0$. This fact can be proven via the use of Itô's formula together with the BHE (4); see [12, Proposition IV.3.5] for an example. To later simplify the proof of (2), we modify ϕ to obtain a local martingale starting at 0. Namely, define

$$\phi^{(\delta)}(t,x) \coloneqq \phi(t+\delta, x) \qquad \forall t \geq 0, \delta \geq 0, x \in \mathbb{R}.$$

Crucially, $\phi^{(\delta)}$ also satisfies the BHE in (4). We summarize the above discussion in the next theorem and include a proof for the sake of completeness.

Theorem 2.1. Let $(X_t)_{t\geq 0}$ be a stochastic process in $\mathbb{R}^n$ such that $(X_{t,i})_{t\geq 0}$ is a standard Brownian motion for each $i \in [n]$ and let $\delta > 0$. Then $(\phi^{(\delta)}(t,X_{t,i}))_{t\geq 0}$ is a local martingale for each $i \in [n]$. Consequently, we have that $\sum_{i=1}^n \phi^{(\delta)}(t,X_{t,i})$ is a local martingale.

Proof. Fix $i \in [n]$ and $t \geq 0$. By Itô's formula [12, Theorem IV.3.3], we have

$$\phi^{(\delta)}(t, X_{t,i}) - \underbrace{\phi^{(\delta)}(0, X_{0,i})}_{=-\sqrt{\delta}} = \int_0^t \partial_x \phi^{(\delta)}(s, X_{s,i}) \, \mathrm{d}X_{s,i} + \int_0^t \underbrace{\partial_t \phi^{(\delta)}(s, X_{s,i}) + \frac{1}{2}\partial_{xx}\phi^{(\delta)}(s, X_{s,i})}_{=0} \mathrm{d}s.$$

Since $\partial_x \phi^{(\delta)}(t, x)$ for any $x \in \mathbb{R}$ and $X_{t,i}$ are both continuous on $t \in [0, +\infty)$, we have that the process $(\int_0^t \partial_x \phi^{(\delta)}(s, X_{s,i}) \, \mathrm{d}X_{s,i})_{t \geq 0}$ is a local martingale that vanishes at 0 [12, Proposition IV.2.10]. Thus, $(\phi^{(\delta)}(t, X_t))_{t \geq 0}$ is a local martingale. $\square$

Remarkably, note that we do not require $(X_t)_{t \geq 0}$ to be an n-dimensional Browninan motion: the above theorem only requires each coordinate of $(X_t)_{t \geq 0}$ to be a Brownian motion, but we do not make any assumptions on the covariation $[X_i, X_j]$ for $i \neq j$. Based on the above theorem, we define our potentials $\Phi^{(\delta)}$ by

$$\Phi^{(\delta)}(t, x) \coloneqq \frac{1}{\eta} \sum_{i=1}^{n} \phi^{(\delta)}(t, x_i) \qquad \forall t \geq 0, \delta \geq 0, x \in \mathbb{R}^n, \tag{5}$$

where $\eta > 0$ is some fixed constant. By Theorem 2.1 we know that $(\Phi^{(\delta)}(t, X_t))_{t \geq 0}$ is a local martingale. To prove (2), we want the potential $\Phi^{(\delta)}(t, x)$ to upper bound $f(t, x)$ modulo some slack that goes to 0 as δ goes to 0. For that, we need to properly define η and β, and both are going to depend on the function λ that appeared in (2). We now formally define λ here; it is effectively the inverse of the function $\phi(1, \cdot)$.

Definition 2.2. For $\alpha \in \mathbb{R}_{\geq 0}$, let $\lambda(\alpha) > 0$ be the unique positive solution to the equation

$$\alpha = -M_0(\lambda(\alpha)^2/2). \tag{6}$$

The existance and uniqueness of the solution of (6) follows directly from item (i) of Lemma 5.2 with $t = 1$. Asymptotic behavior of lower bounds on λ were already studied, such as in [11, Proposition 1(b)]. However, in our case we want to study upper bounds of λ or even the asymptotic behavior of λ itself. In the next lemma we give both an upper bound and the asymptotic behavior of $\lambda(n)$ as $n \to \infty$. This result is partially proven in [7] (since they only show an upper bound on the limit) and we defer the proof to Section 5.

Lemma 2.3. Let $n \in \mathbb{R}$ be positive. Then,

$$\lambda(n) \;\leq\; 3 + \sqrt{2 \ln(n+1)}.$$

Moreover, we have

$$\lim_{n \to \infty} \frac{\lambda(n)}{\sqrt{2 \ln n}} \;=\; 1.$$

Finally, we can prove the following lemma showing that, for carefully picked β in (3) and η in (5), the function $\Phi^{(\delta)}$ upper bounds f minus a slack of $\beta\sqrt{\delta}$.

Lemma 2.4. Let $\delta \geq 0$ and set $\beta = \lambda(n-1)$ in (3). Moreover, fix $t > 0$ and, in Eq. (5), define

$$\eta := \partial_x \phi^{(\delta)}(t, \beta\sqrt{t+\delta}) = \partial_x \phi(t, \beta\sqrt{t}) = \sqrt{\frac{\pi}{2}}\,\mathrm{erfi}\left(\frac{\beta}{\sqrt{2}}\right).$$

Then $\Phi^{(\delta)}(t,x) \geq f(t,x) - \beta\sqrt{\delta}$ for all $x \in \mathbb{R}^n$.

Proof. Let $t > 0$ and assume $x_1 \geq x_2 \geq \cdots \geq x_n$ without loss of generality. By Lemma 5.2, we know $\phi^{(\delta)}(t,\cdot)$ is convex and that $\partial_x \phi^{(\delta)}(t,0) = 0$. Thus, the minimum of $\phi^{(\delta)}(t,\cdot)$ is attained at 0. Then, for any $\bar{\alpha} \in \mathbb{R}$ we have

$$\begin{aligned}
\Phi^{(\delta)}(t,x) &\geq \frac{1}{\eta}\phi^{(\delta)}(t,x_1) + \frac{(n-1)}{\eta}\phi^{(\delta)}(t,0) = \frac{1}{\eta}\phi^{(\delta)}(t,x_1) - \frac{(n-1)}{\eta}\sqrt{t+\delta} \\
&\geq \frac{1}{\eta}\phi^{(\delta)}(t,\bar{\alpha}) + \frac{1}{\eta}\partial_x\phi^{(\delta)}(t,\bar{\alpha})(x_1 - \bar{\alpha}) - \frac{(n-1)}{\eta}\sqrt{t+\delta},
\end{aligned}$$

where the last inequality follows from convexity. If we pick $\beta := \lambda(n-1)$ and $\bar{\alpha} := \mathrm{sign}(x_1)\beta\sqrt{t+\delta}$, then the definition of λ in (2.2) yields $\frac{1}{\eta}\phi^{(\delta)}(t,\bar{\alpha}) = \frac{-1}{\eta}\sqrt{t+\delta}M_0(\lambda(n-1)^2/2) = \frac{n-1}{\eta}\sqrt{t+\delta}$ and by our choice of η we have $\frac{1}{\eta}\partial_x\phi^{(\delta)}(t,\bar{\alpha}) = \mathrm{sign}(x_1)$. Putting all of this together and recalling that $|x_1| = \|x\|_\infty$ we have

$$\begin{aligned}
\Phi^{(\delta)}(t,x) &\geq \mathrm{sign}(x_1)(x_1 - \mathrm{sign}(x_1)\beta\sqrt{t+\delta}) = |x_1| - \beta\sqrt{t+\delta} \\
&\geq \|x\|_\infty - \beta\sqrt{t} - \beta\sqrt{\delta} = f(t,x) - \beta\sqrt{\delta}.
\end{aligned}$$

□

The following corollary completes the proof of (2).

Corollary 2.5. For any stopping time τ we have

$$\mathrm{E}\left[\, \|X_\tau\|_\infty \,\right] \leq \lambda(n-1)\cdot \mathrm{E}\left[\, \sqrt{\tau} \,\right] \leq (3 + \sqrt{2\ln n})\,\mathrm{E}\left[\, \sqrt{\tau} \,\right].$$

Proof. Let η and β be as in Lemma 2.4, and let $\delta > 0$. By Theorem 2.1, we have that $(\Phi^{(\delta)}(t,X_t))_{t\geq 0}$ is a local martingale. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a localizing sequence of stopping times for $(\Phi^{(\delta)}(t,X_t))_{t\geq 0}$. Fix $k \in \mathbb{N}$ and let τ be any bounded stopping time. Then, by the optional stopping theorem and Lemma 2.4 with $\beta := \lambda(n-1)$,

$$-\frac{n\sqrt{\delta}}{\eta} = \mathrm{E}\left[\, \Phi^{(\delta)}(0,X_0) \,\right] = \mathrm{E}\left[\, \Phi^{(\delta)}(\tau\wedge\tau_k, X_{\tau\wedge\tau_k}) \,\right] \geq \mathrm{E}\left[\, f(\tau\wedge\tau_k, X_{\tau\wedge\tau_k}) \,\right] - \beta\sqrt{\delta}.$$

Since this holds for all $\delta > 0$, in the limit $\delta \to 0$ we get $\mathrm{E}\left[\, f(\tau\wedge\tau_k, X_{\tau\wedge\tau_k}) \,\right] \leq 0$. By the definition of f in (3) we have $\mathrm{E}\left[\, \|X_{\tau\wedge\tau_k}\|_\infty \,\right] \leq \beta\,\mathrm{E}\left[\, \sqrt{\tau\wedge\tau_k} \,\right]$. Now by Fatou's Lemma, since $\lim_{k\to\infty}\tau_k = \infty$ almost surely, we have

$$\mathrm{E}\left[\, \|X_\tau\|_\infty \,\right] \leq \liminf_{k\to\infty} \mathrm{E}\left[\, \|X_{\tau\wedge\tau_k}\|_\infty \,\right] \leq \liminf_{k\to\infty} \lambda(n-1)\,\mathrm{E}\left[\, \sqrt{\tau\wedge\tau_k} \,\right] \leq \lambda(n-1)\,\mathrm{E}\left[\, \sqrt{\tau} \,\right].$$

The quantitative bound on λ comes from Lemma 2.3. Finally, to extend the above inequality to general stopping times τ, it suffices to apply the claim for the sequence of stopping times $\{\tau\wedge k\}_{k\in\mathbb{N}}$ and apply Fatou's lemma as we did above. □

Finally, we can slightly generalize Theorem 2.1. In the 1-dimensional case (i.e., $n = 1$), we know that $(\phi^{(\delta)}([X]_t, X_t))_{t\geq 0}$ is a local martingale for any continuous martingale $(X_t)_{t\geq 0}$ that vanishes at 0 [12, Proposition IV.3.4], where $([X]_t)_{t\geq 0}$ is the quadratic variation of $(X_t)_{t\geq 0}$. For general n, however, such a general result does not follow since there is no clear choice of what to use as the quadratic variation of a multidimensional process. Yet, when each $(X_{t,i})_{t\geq 0}$ has a smooth enough quadratic variation, we can prove a very similar result.

Theorem 2.6. Let $\delta > 0$. Let $(X_t)_{t\geq 0}$ be a $\mathbb{R}^n$-valued continuous martingale that vanishes at 0, that is, $(X_t)_{t\geq 0}$ is a stochastic process that is continuous in t and a martingale with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$. Moreover, assume that for each $i \in [n]$ there is a stochastic process $(f_i(t))_{t\geq 0}$ such that $(f_i(t))_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and $[X_i]_t(\omega) = \int_0^t f_i(s)^2 \,\mathrm{d}s$ for each $i \in [n]$. Define $\sigma_t \coloneqq \int_0^t \max_{i\in[n]} f_i(s)^2 \,\mathrm{d}s$. Then $(\Phi^{(\delta)}(\sigma_t, X_t))_{t\geq 0}$ is a local supermartingale.

Proof. Since $\phi^{(\delta)}$ is convex on its second argument by Lemma 5.2, we have $\partial_{x_i,x_i}\Phi(t,\cdot) \geq 0$ for any $i \in [n]$ and $t \geq 0$. Thus, for any $t \geq r \geq 0$ we have

$$\begin{aligned}
&\int_r^t \partial_t \Phi^{(\delta)}(\sigma_s, X_s)\,\mathrm{d}\sigma_s + \frac{1}{2}\sum_{i=1}^n \int_r^t \partial_{x_i,x_i}\Phi^{(\delta)}(\sigma_s, X_s)\,\mathrm{d}[X_i]_s \\
&= \int_r^t \partial_t \Phi^{(\delta)}(\sigma_s, X_s)(\max_{k\in[n]} f_k(s))\,\mathrm{d}s + \frac{1}{2}\sum_{i=1}^n \int_r^t \partial_{x_i,x_i}\Phi^{(\delta)}(\sigma_s, X_s) f_i(s)\,\mathrm{d}s \\
&\leq \int_r^t \partial_t \Phi^{(\delta)}(\sigma_s, X_s)(\max_{k\in[n]} f_k(s)^2)\,\mathrm{d}s + \frac{1}{2}\sum_{i=1}^n \int_r^t \partial_{x_i,x_i}\Phi^{(\delta)}(\sigma_s, X_s)(\max_{k\in[n]} f_k(s)^2)\,\mathrm{d}s \\
&= \int_r^t \sum_{i=1}^n \frac{1}{\eta}\Big(\partial_t \phi^{(\delta)}(\sigma_s, X_{s,i}) + \partial_{xx}\phi^{(\delta)}(\sigma_s, X_{s,i})\Big)(\max_{k\in[n]} f_k(s))\,\mathrm{d}s = 0,
\end{aligned}$$

where the last equation follows since $\phi^{(\delta)}$ satisfies the backwards-heat equation (4). Therefore, from Itô's formula combined with the above inequality we have, for any $t \geq r \geq 0$,

$$\Phi^{(\delta)}(\sigma_t, X_t) \leq \Phi^{(\delta)}(\sigma_r, X_r) + \sum_{i=1}^n \int_r^t \partial_{x_i}\Phi^{(\delta)}(\sigma_s, X_s)\,\mathrm{d}X_{s,i}.$$

Since $\partial_{x_i}\Phi^{(\delta)}$ is continuous, the above stochastic integrals are local martingales. Therefore, $(\Phi^{(\delta)}(\sigma_t, X_t))_{t\geq 0}$ is a local supermartingale. □

As in Theorem 2.1, note that the above theorem does not make any assumptions on $[X_i, X_j]$ for $i \neq j$. Finally, from the last result we get the following generalization of Corollary 2.5. Since its proof is nearly identical, we omit it.

Corollary 2.7. Let $(X_t)_{t\geq 0}$ and σ_t be as in Theorem 2.6. Let τ be a stopping-time for the martingale $(X_t)_{t\geq 0}$. Then,

$$\mathrm{E}[\,\|X_\tau\|_\infty\,] \leq \lambda(n-1)\cdot \mathrm{E}[\,\sqrt{\sigma_\tau}\,] \leq (3 + \sqrt{2\ln n})\,\mathrm{E}[\,\sqrt{\sigma_\tau}\,].$$

In Section 4 we give lower bounds on K_n that show, in some sense, that the results in Corollary 2.5 are nearly tight. Thus, it is natural to wonder whether the bounds in the above corollary are nearly tight as well. However, in this case it is not clear that σ_t is the right quantity to compare $\mathrm{E}[\,\|X_t\|\,]_\infty$ against. The anonymous referee suggested that the choice of σ_t might not be optimal (at least when the entries of X_t are i.i.d.) and one might obtain improved bounds by modifying Φ for this case. We believe this is an interesting question, and we could not find a brief satisfactory answer to this question.

3. Discrete-time Martingales

In this section we prove a similar statement to the one from the previous sections but for a discrete martingale $(S_t)_{t\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $|S_{t,i} - S_{t-1,i}| \leq 1$ for all $i \in [n]$. Namely, we show that for any stopping time τ of the martingale $(S_t)_{t\in\mathbb{N}}$ we have

$$\mathrm{E}[\,\|S_\tau\|_\infty\,] \leq \lambda(n-1)\,\mathrm{E}\left[\,\sqrt{\tau}\,\right] \leq (3+\sqrt{2\ln n})\,\mathrm{E}[\,\sqrt{\sigma_\tau}\,]. \tag{7}$$

Interestingly, the proof technique we use here mimics the one used for continuous-time martingales via the use of a discrete version of Itô's formula [8], and we easily get true (super)martingales instead of the localized versions. In order to do so, we shall make use of *discrete derivatives* in a way similar to [6]. For any bivariate function g, define its discrete derivatives as

$$\begin{aligned} g_t(t,x) &= g(t,x) - g(t-1,x), \\ g_x(t,x) &= \frac{g(t,x+1) - g(t,x-1)}{2}, \\ g_{xx}(t,x) &= (g(t,x+1) + g(t,x-1)) - 2g(t,x). \end{aligned} \tag{8}$$

A key property of ϕ is that, similar to the continuous-time case, the potential function ϕ satisfies a discrete version of the backwards-heat equation as an inequality. This was already shown by [7], but we include a short proof of this fact using properties of M_0 for the sake of completeness. Similar to the results from the previous section, this discrete backwards-heat inequality will be key to show that $(\Phi(t,S_t))_{t\geq 1}$ is a supermartingale.

Lemma 3.1. For any $t \in \mathbb{N}$ with $t > 1$ and $x \in \mathbb{R}$, we have $\phi_t(t,x) + \frac{1}{2}\phi_{xx}(t,x) \leq 0$.

Proof. Let $t > 1$ and $x \in \mathbb{R}$. Then the statement of the lemma is equivalent to

$$\sqrt{t}\cdot M_0\left(\frac{(x+1)^2}{2t}\right) + \sqrt{t}\cdot M_0\left(\frac{(x-1)^2}{2t}\right) \geq 2\sqrt{t-1}\cdot M_0\left(\frac{x^2}{2(t-1)}\right). \tag{9}$$

Note that $t > 1$, so all terms are well-defined. Rearranging, this is equivalent to

$$M_0\left(\frac{(x+1)^2}{2t}\right) + M_0\left(\frac{(x-1)^2}{2t}\right) \geq 2\sqrt{1-1/t}\cdot M_0\left(\frac{x^2}{2t}\right)$$

which follows from Lemma 5.6 by setting x and z in Lemma 5.6 to $x/\sqrt{t}$ and $1/\sqrt{t}$, respectively. $\square$

Furthermore, we shall make use of the following discrete version of Itô's formula for convex functions. This version of the lemma was stated in [7], but it follows easily from previous results (e.g., see [8, Example 10.9] and [6, Lemma 3.13]).

Lemma 3.2 ([7, Lemma 4.7])**.** Let $x_1, x_2, \cdots$ be a sequence of real numbers such that $|x_t - x_{t-1}| \leq 1$ and let f be a bivariate function that is convex on its second argument. Then for any integer $T \geq 2$, we have

$$f(T, x_T) - f(1, x_1) \leq \sum_{t=2}^{T} f_x(t, x_{t-1}) \cdot (x_t - x_{t-1}) + \sum_{t=2}^{T} \left(\frac{1}{2} f_{xx}(t, x_{t-1}) + f_t(t, x_{t-1}) \right).$$

With both of these lemmas we can show that $(\Phi(t, S_t))_{t \geq 1}$ is a discrete-time supermartingale. Actually, that fact can be established without using the discrete Itô's formula. Nevertheless, using the discrete Itô's formula guides us in the claims needed to show that $(\Phi(t, S_t))_{t \geq 1}$ is a supermartingale.

Theorem 3.3. Let $(S_t)_{t \geq 1}$ be a discrete-time martingale in $\mathbb{R}^n$ such that $|S_{t,i} - S_{t-1,i}| \leq 1$ for any $i \in \mathbb{N}$. Then $(\Phi(t, S_t))_{t \geq 1}$ is a discrete-time supermartingale.

Proof. Note that it suffices to show that $(\phi(t, S_{t,i}))_{t \geq 1}$ is a supermartingale for each $i \in [n]$ to show that $(\Phi(t, S_t))_{t \geq 1}$ is a supermartingale. Thus, we may assume $n = 1$ and show that $(\phi(t, S_t))_{t \geq 1}$ to prove the statement of the theorem.

Since S_t is bounded for any fixed $t \geq 1$, we have that $\phi(t, S_t)$ is integrable for any fixed $t \geq 1$. Now let $t \in \mathbb{N}$ with $t > 1$. It only remains to show that $\mathrm{E}[\, \phi(t, S_t) \mid \mathcal{F}_{t-1} \,] \leq \phi(t-1, S_{t-1})$ for $\mathcal{F}_{t-1} \coloneqq \sigma(S_1, \ldots, S_{t-1})$. By Lemma 3.2, we have $\phi(t, S_t) - \phi(t-1, S_{t-1})$ is upper bounded by

$$\phi_x(t, S_{t-1}) \cdot (S_t - S_{t-1}) + \frac{1}{2} \phi_{xx}(t, S_{t-1}) + \phi_t(t, S_{t-1}).$$

For the first term above, note that since $(S_t)_{t \geq 1}$ is a martingale,

$$\mathrm{E}[\, \phi_x(t, S_{t-1})(S_t - S_{t-1}) \mid \mathcal{F}_{t-1} \,] = \mathrm{E}[\, \phi_x(t, S_{t-1})(S_{t-1} - S_{t-1}) \mid \mathcal{F}_{t-1} \,] = 0.$$

This fact together with the discrete backwards-heat inequality from Lemma 3.1 yields

$$\begin{aligned}
&\mathrm{E}[\, \phi(t, S_t) - \phi(t-1, S_{t-1}) \mid \mathcal{F}_{t-1} \,] \\
&\leq \mathrm{E}\left[\phi_x(t, S_{t-1}) \cdot (S_t - S_{t-1}) + \frac{1}{2} \phi_{xx}(t, S_{t-1}) + \phi_t(t, S_{t-1}) \mid \mathcal{F}_{t-1} \right] \\
&\leq \frac{1}{2} \phi_{xx}(t, S_{t-1}) + \phi_t(t, S_{t-1}) \leq 0
\end{aligned}$$

Therefore, $(\phi(t, S_t))_{t \geq 1}$ is a supermartingale. □

We are now in position to prove (7), which easily follows from the previous theorem.

Corollary 3.4. Let $(S_t)_{t \geq 0}$ be a martingale in $\mathbb{R}^n$ with $S_0 = 0$ and $|S_{t,i} - S_{t-1,i}| \leq 1$ almost surely. Then, for any stopping time τ we have

$$\mathrm{E}[\, \|S_\tau\|_\infty \,] \leq \lambda(n-1)\, \mathrm{E}\left[\sqrt{\tau} \right] \leq (3 + \sqrt{2 \ln n})\, \mathrm{E}\left[\sqrt{\tau} \right].$$

Proof. Let us prove the statement of the corollary for a bounded stopping time τ. The extension to general stopping times follows from Fatou's lemma in a way identical to the use of Fatou's lemma in the proof of Corollary 2.5.

Let η and β be as in Lemma 2.4. Then, $\Phi(t,x) \geq f(t,x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Furthermore, by Theorem 3.3 we have that $(\Phi(t,S_t))_{t\geq 1}$ is a supermartingale. Thus, using the optional stopping theorem together with the facts that $\phi(1,\cdot)$ is increasing in $[0,+\infty)$ and that $\lambda(0) > 1 \geq |S_{1,i}|$ for any $i \in [n]$ we have

$$0 = \Phi(1, \lambda(0)\cdot \mathbb{1}) \geq \Phi(1,\mathbb{1}) \geq \mathrm{E}\left[\,\Phi(1,S_1)\,\right] \geq \mathrm{E}\left[\,\Phi(\tau,S_\tau)\,\right] \geq \mathrm{E}\left[\,f(\tau,S_\tau)\,\right],$$

which implies the first desired inequality. The bound on $\lambda(n-1)$ follows from Lemma 2.3. $\square$

Note that the above result only requires the martingales increments to be bounded.[1] Davis [2] proves similar results for 1-dimensional martingales from his continuous time results via the use of Skorohod embeddings. While his results do not assume bounded increments, the results assume the increments have distributions symmetric around 0.

Although Theorem 3.3 assumes the increments of S_t to be in $[-1,1]$, we should expect the same bounds to hold with increments in an interval $[-\alpha,\beta]$ with $\alpha,\beta > 0$ as long as $\alpha+\beta \leq 2$. However, if $\max\{\alpha,\beta\} > 1$, then we would need to scale S_t to apply the theorem. Yet, we conjecture that even in this case $(\phi(t,S_t))_{t\geq 1}$ is indeed a supermartingale. This in fact follows from Conjecture 5.7, which can be seen as a generalization of the discrete backwards-heat inequality

4. Lower Bound on K_n

One can get an easy lower bound on K_n with the right asymptotic behavior in n from classical lower bounds on $\mathrm{E}\left[\max_{i\in[n]}|g_i|\right]$ where $g_1,\ldots,g_n$ are i.i.d. standard Gaussian random variables. Namely, one can show $K_n \geq (1-\varepsilon_n)\sqrt{2\ln n}$ with $\varepsilon_n > 0$ going to 0 as n grows [10, Theorem 3]. In this section, let us show a slightly better lower bound c_n on K_n that is also asymptotically tight. For that, we shall use a few confluent hypergeometric functions that are different from M_0. Namely, for any $a,b \in \mathbb{R}$ with $b \notin \mathbb{Z}_{\leq 0}$, the confluent hypergeometric function of the first kind (with parameters a and b) is defined as

$$M(a,b,x) \coloneqq \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!}, \qquad \forall x \in \mathbb{R},$$

where $(z)_k \coloneqq \prod_{i=1}^{k-1}(x+i)$ is the Pochhammer symbol. Other characterizations and properties of confluent hypergeometric functions can be found in [9, §13].

For each n, let $c_n > 0$ be the smallest positive root of

$$x \in \mathbb{R} \mapsto M\Big(-\frac{1}{2n}, \frac{1}{2}, \frac{x^2}{2}\Big).$$

[1]The corollary requires the increments to be bounded by 1, yet one can scale the martingale to allow for increments bounded by arbitrary constants.

Such c_n exists for any $n > 0$ since it is known that $M(a, b, \cdot)$ has $\lceil -a \rceil$ positive roots if $a < 0$ and $b \geq 0$ [9, §13.9(i)]. The lower bound in the next theorem follows from a careful choice of a stopping time for a standard Browninan motion. One may find it interesting that such a tight lower bound arises from a martingale with independent coordinates.

Theorem 4.1. Let $n \geq 1$ and K_n be defined as in (1). Then $c_n \leq K_n$.

Proof. Fix $n \geq 1$. We shall show that $c_n - \varepsilon \leq K_n$ for all $\varepsilon > 0$, implying that $c_n \leq K_n$. Let $\varepsilon > 0$ and set $c' := c_n - \varepsilon$. Also, assume ε is sufficiently small so that $c' > 0$.

Let $(B_t)_{t \geq 0}$ be an n-dimensional Brownian motion. For each $i \in [n]$, define

$$\tau_i := \inf \left\{ t \geq 0 \,:\, |B_{i,t}| > c' \sqrt{t+1} \right\}$$

and set $\tau := \min_{i \in [n]} \tau_i$. Note that by definition of τ, if $\mathrm{E}[\sqrt{\tau}] < \infty$ then $\mathrm{E}[\|B_\tau\|_\infty] \geq c' \cdot \mathrm{E}\left[\sqrt{\tau+1}\right] > c' \cdot \mathrm{E}[\sqrt{\tau}]$. Thus, let us show that $\sqrt{\tau}$ is integrable.

Following Perkins' notation from [11], for any $c > 0$ let $\lambda_0(-c, c)$ be such that c is the smallest positive root of $x \mapsto M(-\lambda_0(-c, c), 1/2, x^2/2)$. By the choice of c_n we have $\lambda_0(-c_n, c_n) = 1/2n$. On top of that, by [11, Proposition 1] we know that $\lambda(-c, c)$ is strictly decreasing for $c \in (0, \infty)$. Thus, since $0 < c' < c_n$, we have $\lambda(-c', c') \in (\frac{1}{2n}, 1)$. Finally, by the tail bound from [11, Lemma 10.(a)], there is a constant $C > 0$ such that $\Pr[\tau_i > t] \leq C(t+1)^{-\lambda(-c', c')}$. Therefore, since $-2n\lambda(-c', c') < -1$ we have

$$\begin{aligned}
\mathrm{E}\left[\sqrt{\tau}\right] &= \int_0^\infty \Pr\left[\sqrt{\tau} > s\right] \mathrm{d}s = \int_0^\infty \prod_{i=1}^n \Pr\left[\tau_i > s^2\right] \mathrm{d}s \\
&\leq C \int_0^\infty (s+1)^{-2n\lambda(-c', c')} \,\mathrm{d}s < \infty.
\end{aligned}$$

□

Finally, on the asymptotic behavior of c_n, we know from [11, Proposition 1.(b)] that

$$\lim_{n \to \infty} \frac{1}{2n} \cdot \frac{\exp(c_n^2/2)}{c_n} = \frac{1}{2\pi}.$$

Therefore, we know $c_n \sim \sqrt{2 \ln n}$ and that for n big enough we have $c_n > \sqrt{2 \ln n}$. This means that c_n yields a better lower bound than the lower bound we have on $\mathrm{E}[\|B_t\|_\infty]$ for fixed t. Moreover, this also tells us that $\lambda(n-1) - c_n \leq 3$ for sufficiently large n. It may be that $c_n = K_n$, but we were not able to prove this result and leave it as an open question.

5. Properties of the Confluent Hypergeometric Function

In this section we outline some of the main properties of the confluent hypergeometric function M_0 that we use throughout this note. All the properties in this section can be found (or be easily derived from) [6, Section 2.6] and [7]. Yet, we present the proof of some of these properties for the sake of completeness.

Fact 5.1 ([6, Facts 2.4, 2.5, and 2.6]). We have

(i) $M_0'(x) = -\frac{\sqrt{\pi}}{2\sqrt{x}}\operatorname{erfi}(\sqrt{x})$ for all $x > 0$ and $M_0'(0) = 0$;

(ii) $M_0(x)$ is strictly decreasing and concave on $[0, +\infty)$.

The above properties of M_0 allow us to derive many properties about the function $\phi(t, x) = -\sqrt{t} \cdot M_0(x^2/2t)$, as we show in the next lemma.

Lemma 5.2. Let $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}$. Then

(i) $\phi(t, \cdot)$ is convex on $\mathbb{R}$, strictly increasing on $[0, +\infty)$, and its image is $[-\sqrt{t}, +\infty)$;

(ii) $\partial_x \phi(t, x) = \sqrt{\frac{\pi}{2}}\operatorname{erfi}(x/\sqrt{2t})$;

(iii) $\partial_{xx}\phi(t, x) = \frac{1}{\sqrt{t}}\exp(x^2/2t)$ for $x > 0$;

(iv) $\partial_t \phi(t, x) = -\frac{1}{2\sqrt{t}}\exp(x^2/2t)$;

Proof. For (i), note the from (5.1) we know $M_0(0) = 1$ and that M_0 is concave and strictly decreasing on $[0, +\infty)$ and its image over this domain is $(-\infty, 1]$ (since its derivative is negative and strictly decreasing). For property (ii) we have, by the chain-rule and Fact 5.1, that $\partial_x \phi(t, x)$ is $\operatorname{sign}(x)\sqrt{\frac{\pi}{2}}\operatorname{erfi}(|x|/\sqrt{2t}) = \sqrt{\frac{\pi}{2}}\operatorname{erfi}(x/\sqrt{2t})$ for $x \neq 0$ and 0 for $x = 0$. Property (iii) follows from the fundamental theorem of calculus together with the chain rule since $\sqrt{\frac{\pi}{2}}\operatorname{erfi}(x) = \sqrt{2}\int_0^x e^{z^2}\,\mathrm{d}z$. For (iv), assume for notational simplicity only that $x > 0$. Then,

$$\begin{aligned}
\partial_t\left(\sqrt{t}M_0\left(\frac{x^2}{2t}\right)\right) &= \frac{1}{2\sqrt{t}}M_0\left(\frac{x^2}{2t}\right) - \frac{x^2}{2t^{3/2}}M_0'\left(\frac{x^2}{2t}\right) \\
&= \frac{1}{2\sqrt{t}}\left(\exp\left(\frac{x^2}{2t}\right) - \sqrt{\pi}\frac{x}{\sqrt{2t}}\operatorname{erfi}\left(\frac{x}{\sqrt{2t}}\right) - \frac{x^2}{t}\frac{\sqrt{\pi 2t}}{2x}\operatorname{erfi}\left(\frac{x}{\sqrt{2t}}\right)\right) \\
&= \frac{1}{2\sqrt{t}}\exp\left(\frac{x^2}{2t}\right).
\end{aligned}$$

□

Fact 5.3 ([9, Section 7.8])**.** For all $z > 0$, we have

$$\frac{\sqrt{\pi}}{2}\operatorname{erfi}(z) = \int_0^z e^{t^2}\,\mathrm{d}t < \frac{e^{z^2} - 1}{z}. \tag{10}$$

Lemma 5.4. For every $x \in \mathbb{R}$, we have $1 - M_0(x^2/2) < \exp(x^2/2) - 1$.

Proof. Using Facts 5.1 and 5.3, for any $x \neq 0$ we have

$$\begin{aligned}
1 - M_0(x^2/2) &= 1 - \exp(x^2/2) + \sqrt{\frac{\pi}{2}}|x|\operatorname{erfi}(|x|/\sqrt{2}) \\
&< 1 - \exp(x^2/2) + \sqrt{2}|x|\frac{\exp(x^2/2) - 1}{|x|/\sqrt{2}} \\
&= \exp(x^2/2) - 1.
\end{aligned}$$

□

The next lemma from [7] gives an upper bound to $M_0(x^2/2)$ for $x \geq 0$, and we add a proof for the sake of completeness. Our main use of this lemma will be to bound the values of $\lambda(n)$ for $n \geq 0$.

Lemma 5.5 ([7, Lemma A.2]). For every $x \geq 0$,

$$1 - M_0(x^2/2) \;\geq\; \frac{\exp(x^2/2)}{x^2 + 1 + 2/x^2} \qquad \forall x \geq 0.$$

Proof. Define $f(x) = 1 - M_0(x^2/2)$ and $g(x) = \exp(x^2/2)/(x^2 + 1 + 2/x^2)$. The derivatives are

$$f'(x) = \sqrt{\pi/2}\,\mathrm{erfi}(x/\sqrt{2}) \quad \text{and} \quad g'(x) = \exp(x^2/2) \cdot \frac{x^7 - x^5 + 2x^3 + 4x}{(x^4 + x^2 + 2)^2}.$$

The second derivatives are

$$\begin{aligned} f''(x) &= \exp(x^2/2) \qquad \text{and} \\ g''(x) &= \exp(x^2/2) \cdot \frac{x^{12} - x^{10} + 9x^8 + 7x^6 - 32x^4 + 8x^2 + 8}{(x^4 + x^2 + 2)^3}. \end{aligned}$$

We will show that $f''(x) \geq g''(x)$. By rearranging, this amounts to showing that

$$x^{12} - x^{10} + 9x^8 + 7x^6 - 32x^4 + 8x^2 + 8 \;\leq\; (x^4 + x^2 + 2)^3. \tag{11}$$

Expanding the right-hand side, we get

$$(x^4 + x^2 + 2)^3 \;=\; x^{12} + 3x^{10} + 9x^8 + 13x^6 + 18x^4 + 12x^2 + 8.$$

The right-hand side coefficients are no smaller than the ones on the left-hand side of (11), which shows that $f''(x) \geq g''(x)$ for $x \geq 0$.

By integrating and using that $f'(0) = g'(0) = 0$, we obtain that $f'(x) \geq g'(x)$ for all $x \geq 0$. Finally, by integrating again and using that $f(0) = g(0) = 0$, we obtain $f(x) \geq g(x)$ for all $x \geq 0$. □

With the above lemma we can present the proof of Lemma 2.3. We note that this result is already partially proven in [7, Lemma A.5] but here we show that the limit is exactly equal to 1 (instead of only bounded by 1).

Proof. (of Lemma 2.3) Define $\ell = 3 + \sqrt{2\ln(n+1)}$. Note that $\ell^2 = 2\ln(n+1) + 6\ell - 9$. So, by Lemma 5.5,

$$1 - M_0(\ell^2/2) \;\geq\; \frac{\exp(\ell^2/2)}{\ell^2 + 1 + 2/\ell^2} \;=\; \frac{\exp(\ln(n+1) + 3\ell - 9/2)}{\ell^2 + 1 + 2/\ell^2}. \tag{12}$$

Since $\ell \geq 3$ we have $3\ell - 9/2 \geq \ell$, and also

$$e^{\ell} \;\geq\; \ell^2 + 1 + 2/\ell. \tag{13}$$

This may be seen by a direct calculation for $\ell = 3$, then observing that the derivative of the left-hand side exceeds the derivative of the right-hand side for $\ell \geq 3$. Combining (12) and (13) we obtain

$$1 - M_0(\ell^2/2) \;\geq\; \exp(\ln(n+1)) \;=\; n+1 \;=\; -M_0(\lambda(n)^2/2) + 1.$$

Since $-M_0$ is monotonically increasing, it follows that $\lambda(n) \leq \ell$. With that, we already have that

$$\limsup_{n\to\infty} \frac{\lambda(n)}{\sqrt{2\ln n}} \leq 1.$$

Thus, to prove that the limit of $\lambda(n)/\sqrt{2\ln n}$ exists and is equal to 1 it suffices to show that $\lambda(n) \geq \sqrt{2\ln(n+1)}$ for all $n \geq 1$. The latter follows similarly as above from Lemma 5.4 since, for $\ell := \sqrt{2\ln(n+1)}$ we have,

$$1 - M_0(\ell^2/2) < \exp(\ell^2/2) = n+1 = 1 - M_0(\lambda(n)^2/2),$$

which shows that $\sqrt{2\ln(n+1)} = \ell \leq \lambda(n)$. □

The next lemma from [6, Lemma 3.10] is used to prove that the hypergeometric potential satisfies the discrete version of the backwards heat inequality.

Lemma 5.6 ([6, Lemma 3.10])**.** For all $z \in [0,1)$ and $x \in \mathbb{R}$, we have

$$M_0\left(\frac{(x+z)^2}{2}\right) + M_0\left(\frac{(x-z)^2}{2}\right) \geq 2\sqrt{1-z^2}\,M_0\left(\frac{x^2}{2(1-z^2)}\right).$$

The discrete backwards-heat inequality (Lemma 3.1) follows from the above lemma. With it, we showed in Section 3 that $(\phi(t,S_t))_{t\geq 1}$ is a supermartingale if the increments of the martingale S_t are in $[-1,1]$. We conjecture that the same result should hold even if the increments of S_t are in an arbitrary interval of length 2 that contains 0. This would follow if the following conjecture, an asymetric version of Lemma 5.6, were true.

Conjecture 5.7. For all $q \in [0,2]$, $z \in [0,1)$, and $w \in \mathbb{R}$,

$$\left(1-\frac{q}{2}\right)M_0\left(\frac{(w+zq)^2}{2}\right) + \left(\frac{q}{2}\right)M_0\left(\frac{(w+z(q-2))^2}{2}\right) \;\geq\; \sqrt{1-z^2}\,M_0\left(\frac{w^2}{2(1-z^2)}\right).$$

Declaration of competing interest

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