On the Empirical Time Complexity of Random 3-SAT at the Phase Transition

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Complexity and SAT

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- Scaling of running time as function of instance size

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Propositional satisfiability problem (SAT)

- First problem proved to be $\mathscr{NP}$-complete (Cook, 1971)
- Intense academic interest & many **practical** applications
- Dramatic & sustained progress in SAT solving

Outline

1. Background & Location of Phase Transition
2. Scientific Questions & Related Work
3. Empirical Scaling Analysis – Methodology
4. Empirical Scaling Results
   - DPLL-based Solvers
   - SLS-based Solvers
5. Conclusions

Background – Phase Transition

Soluability phase transition: 50% of random instances satisfiable

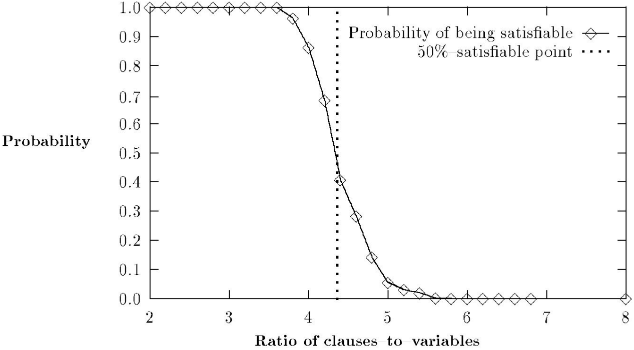

Figure from (Mitchell *et al.*, 1992)

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Soluability phase transition: 50% of random instances satisfiable

- Phase transition is sharp (Cheeseman *et al.*, 1991)
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Widely studied instance distribution

- Prominent model of computational hardness in SAT and beyond
  - For DPLL-based solvers (Mitchell *et al.*, 1992)
  - For SLS-based solvers (Yokoo, 1997)
  - ...

Location of Phase Transition

Best previous model (Crawford and Auton, 1996):

$$m_c = 4.258 \cdot n + 58.26 \cdot n^{-2/3}$$

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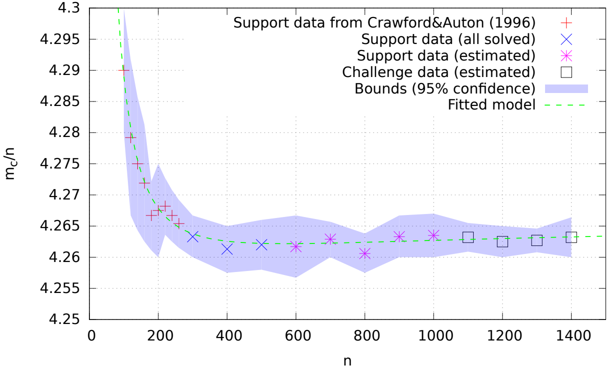
Weaknesses:

- Inconsistent with results from cavity method (Mertens *et al.*, 2006):

$$\lim_{n \to \infty} m_c / n = 4.26675 \pm 0.00015$$

- Under-estimates m_c for larger n

Location of Phase Transition



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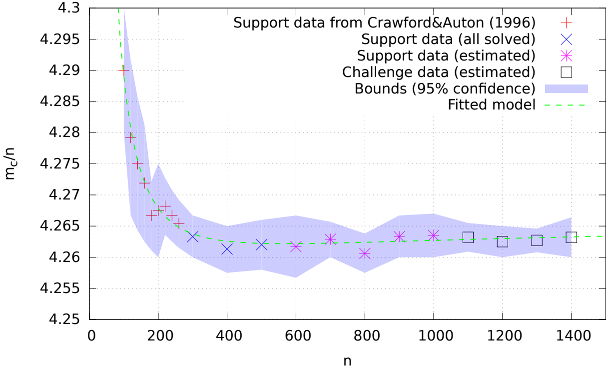

Refined model:

$$m_c = 4.26675 \cdot n + 447.884 \cdot n^{-0.0350967} - 430.232 \cdot n^{-0.0276188}$$

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Solvers studied:

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Solvers studied:

- DPLL-based: kcnfs, march_hi, march_br
- SLS-based: WalkSAT/SKC, BalancedZ, probSAT

Related Work

Observations on empirical scaling of:

- SLS-based solvers, e.g., Gent and Walsh (1993); Gent *et al.* (1997)
- DPLL-based solvers, e.g., Coarfa *et al.* (2003)

Related Work

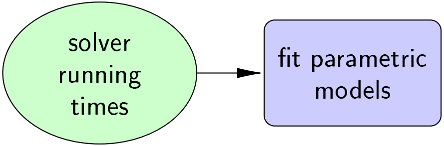
Observations on empirical scaling of:

- SLS-based solvers, e.g., Gent and Walsh (1993); Gent *et al.* (1997)
- DPLL-based solvers, e.g., Coarfa *et al.* (2003)

Limitations:

- # variable flips vs. actual running times,
  e.g., Gent and Walsh (1993); Gent *et al.* (1997)
- Inconclusive results, e.g., Gent and Walsh (1993)
- Simple curve fitting & vague definition of "good fit"

Methodology (Hoos, 2009; Hoos and Stützle, 2014)



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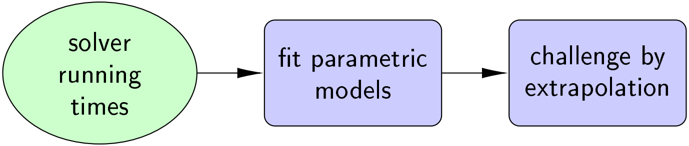

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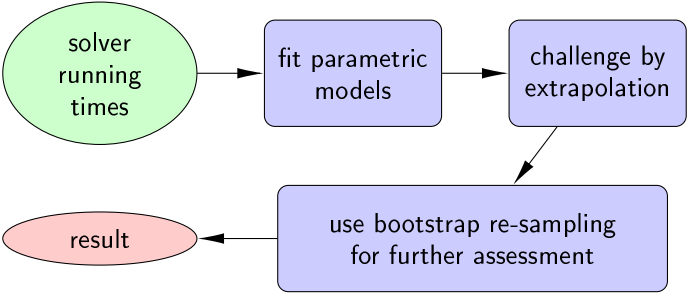

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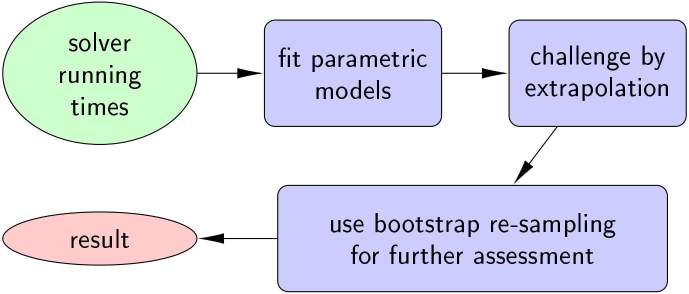


Extensions made in this work:

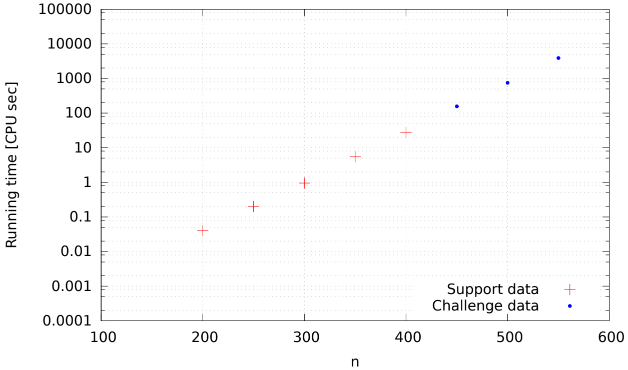
- Use confidence intervals of observed data to assess models
- Compare scaling models of two solvers based on confidence intervals of observed/predicted data

Empirical Scaling Results – DPLL-based Solvers

Divide instance sets into support and challenge:

n	200	250	300	350	400
median	0.040	0.200	0.950	5.455	27.580

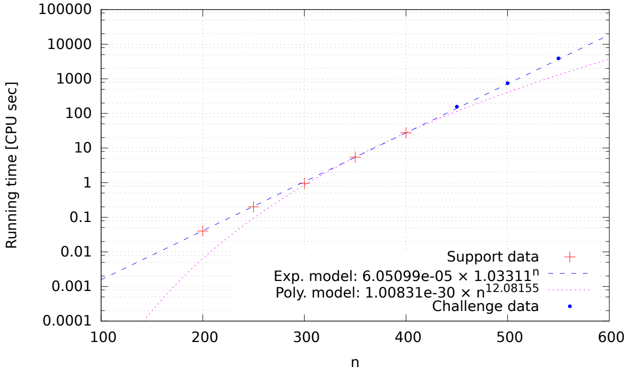
n	450	500	550
median	156.480	750.510	3896.450



Empirical Scaling Results – DPLL-based Solvers

Fit parametric models:

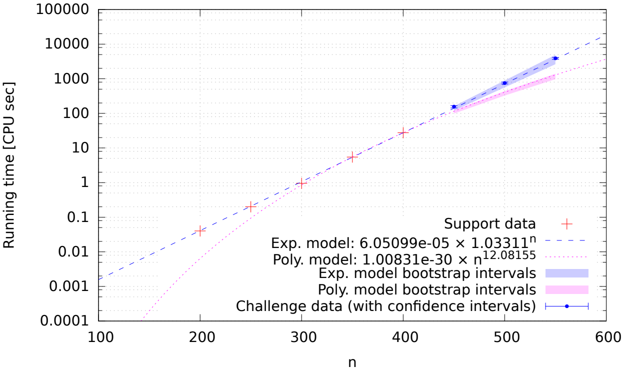
		Model	RMSE (support)	RMSE (challenge)
kcnfs	**Exp. Model**	$\mathbf{4.30400 \times 10^{-5} \times 1.03411^{n}}$	**0.05408**	**143.3**
	Poly. Model	$9.40745 \times 10^{-31} \times n^{12.1005}$	0.06822	1516



Empirical Scaling Results – DPLL-based Solvers

Bootstrap re-sampling:

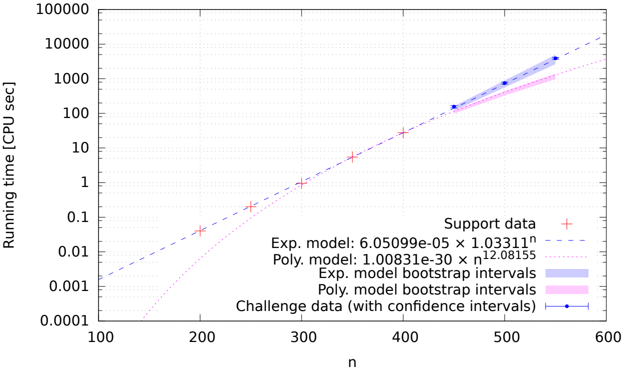
Solver	n	Predicted confidence intervals		Observed median run-time (sec)	
		Poly. model	Exp. model	Point estimates	Confidence intervals
kcnfs	450	[98.326, 122.115]	**[120.078, 161.444]**	156.480	[143.340, 166.770]
	500	[327.997, 439.089]	**[561.976, 889.428]***	750.510	[708.290, 806.130]
	550	[971.862, 1402.255]	**[2622.488, 4901.661]***	3896.450	[3633.630, 4130.915]



Empirical Scaling Results – DPLL-based Solvers

Bootstrap re-sampling:

Solver	Model	Confidence interval of a	Confidence interval of b
kcnfs	Poly.	$[3.33969 \times 10^{-31}, 4.30846 \times 10^{-29}]$	$[11.4234, 12.2674]$
	Exp.	$[3.33378 \times 10^{-5}, 1.07425 \times 10^{-4}]$	$[1.03136, 1.03476]$

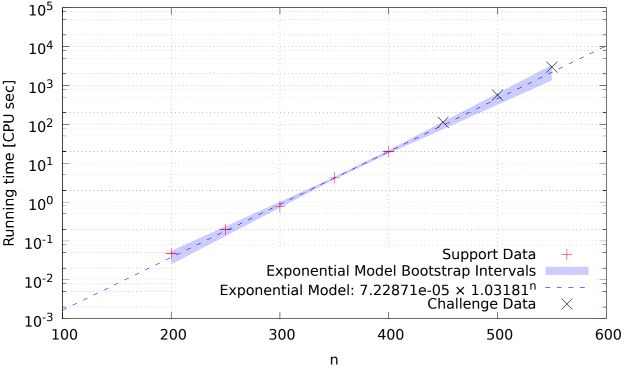


Empirical Scaling Results – DPLL-based Solvers

Compare scaling models:

- No significant difference between two march-variants
- Two march-variants scale significantly better than kcnfs

Scaling models of march_hi:



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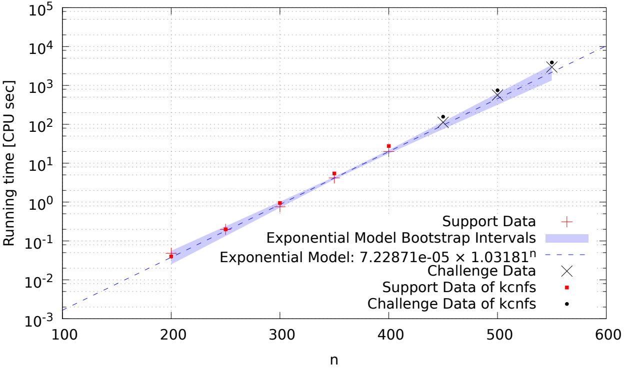
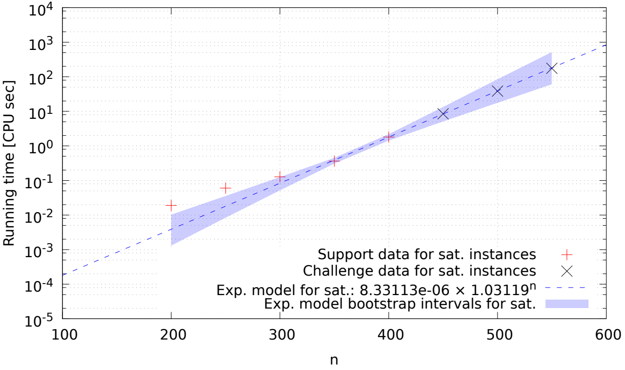
Compare scaing models of kcnfs against march_hi:



Empirical Scaling Results – DPLL-based Solvers

Difference in solving satisfiable instances and unsatisfiable instances:



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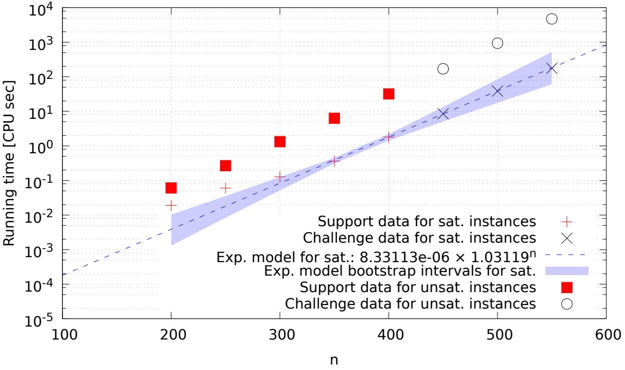

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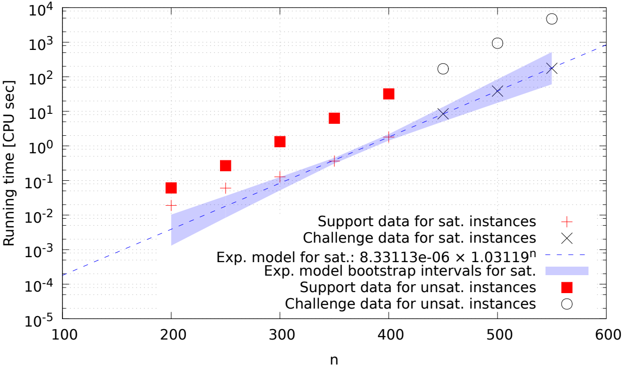
Difference in solving satisfiable instances and unsatisfiable instances:

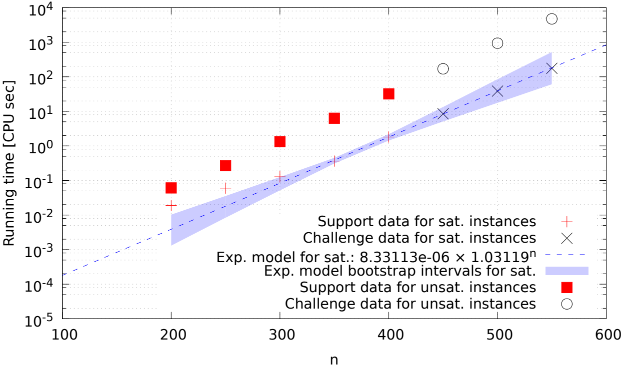
- Is the difference a constant factor?



Empirical Scaling Results – DPLL-based Solvers

Difference in solving satisfiable instances and unsatisfiable instances:

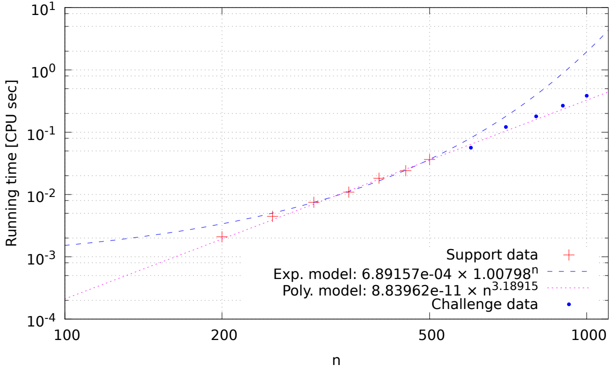
- Is the difference a constant factor?
- Fit running times of solving unsatisfiable instances with model $a \cdot b_{\text{sat}}^n$
  - Slower in solving unsatisfiable instances by constant factor only



Empirical Scaling Results – SLS-based Solvers

Fit parametric models:

		Model	RMSE (support)	RMSE (challenge)
WalkSAT/SKC	Exp. Model	$6.89157 \times 10^{-4} \times 1.00798^{n}$	0.0008564	0.7600
	Poly. Model	$\mathbf{8.83962 \times 10^{-11} \times n^{3.18915}}$	**0.0007433**	**0.03142**



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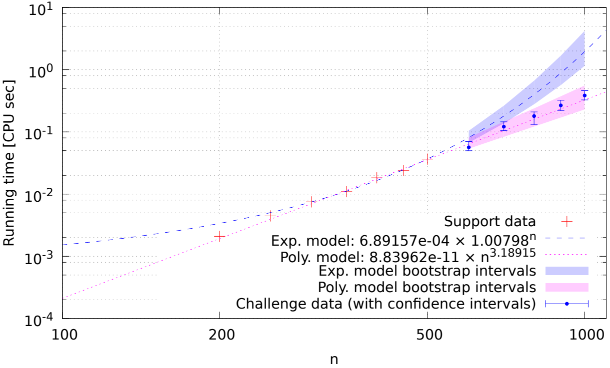
Bootstrap re-sampling:

Solver	n	Predicted confidence intervals		Observed median run-time (sec)	
		Poly. model	Exp. model	Point estimates	Confidence intervals
WalkSAT/SKC	600	**[0.054, 0.081]**	[0.067, 0.104]	0.056	[0.050, 0.070]
	⋮	⋮	⋮	⋮	⋮
	1000	**[0.229, 0.557]***	[1.151, 4.200]	0.385	[0.327, 0.461]



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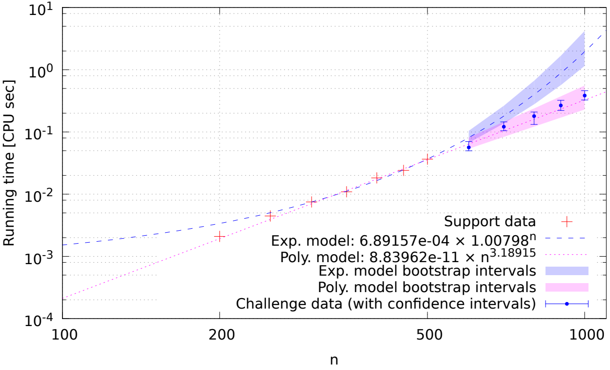
Solver	Model	Confidence interval of a	Confidence interval of b
WalkSAT/SKC	Exp.	$[4.05064 \times 10^{-4}, 1.00662 \times 10^{-3}]$	$[1.00709, 1.00924]$
	Poly.	$[2.58600 \times 10^{-12}, 8.63869 \times 10^{-10}]$	$[2.80816, 3.76751]$



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Higher quantiles:

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Even larger instances:

- Limited experiments on instances of $n \in \{1500, 2000, 5000\}$
- Data consistent with polynomial models

Conclusions

Refined model for location of 3-SAT phase transition

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- How do running times of high-performance SAT solvers scale?
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Methodology applicable to other algorithms, instances and problems

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- ESA: automated tool for scaling analysis (Mu and Hoos, 2015)
  `www.cs.ubc.ca/labs/beta/Projects/ESA/esa-online.html`

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